SPECTRA OF INTEGRABLE QUANTUM MAGNETS VIA CLASSICAL MANY-BODY SYSTEMS

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based on

joint works with A.Alexandrov, A.Gorsky,

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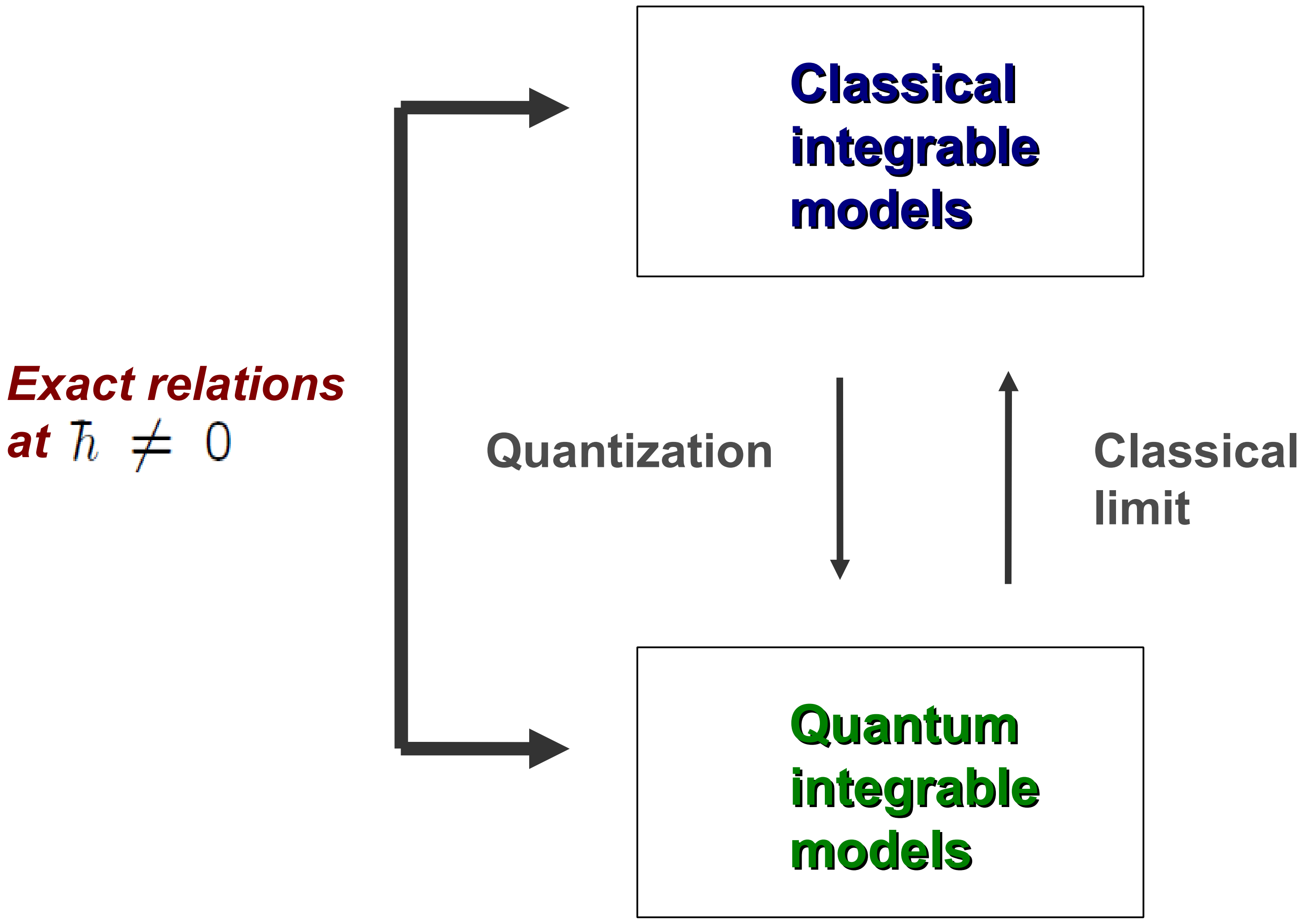


Classical integrable equations in quantum integrable models at $\hbar \neq 0$

Quantum integrable models

Heisenberg XXX spin chain

$$H^{XXX} = \frac{1}{2}\sum_{j=1}^{N}\left(\sigma_1^{(j)}\sigma_1^{(j+1)} + \sigma_2^{(j)}\sigma_2^{(j+1)} + \sigma_3^{(j)}\sigma_3^{(j+1)} - 1\right)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad N+1 \equiv 1$$

Exact solution (Bethe ansatz)

$$E = \sum_{j=1}^{M} \varepsilon(u_\alpha), \quad \varepsilon(u) = -\frac{4}{1+4u^2}$$

Bethe equations:

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^N = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

A formal parameter η

$$E = \sum_{j=1}^{M} \varepsilon(u_\alpha), \quad \varepsilon(u) = \frac{4}{\eta^2 - 4u^2}$$

Bethe equations:

$$\left(\frac{u_j + \frac{\eta}{2}}{u_j - \frac{\eta}{2}}\right)^N = \prod_{k \neq j}^{M} \frac{u_j - u_k + \eta}{u_j - u_k - \eta}$$

Heisenberg XXX model via the Quantum inverse scattering method

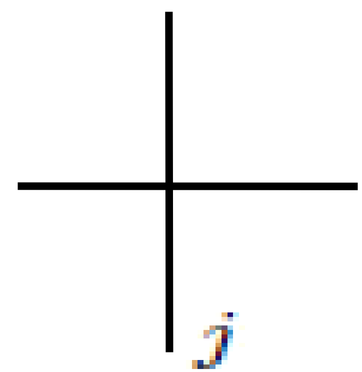
The quantum Lax operator

$$L(x) = \begin{pmatrix} x + \frac{\eta}{2}(1+\sigma_3) & \eta\sigma_- \\ \eta\sigma_+ & x + \frac{\eta}{2}(1-\sigma_3) \end{pmatrix} = xI + \eta P$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The Lax operator on j-th site

$$L_j(x) = \begin{pmatrix} x + \frac{\eta}{2}(1+\sigma_3^{(j)}) & \eta\sigma_-^{(j)} \\ \eta\sigma_+^{(j)} & x + \frac{\eta}{2}(1-\sigma_3^{(j)}) \end{pmatrix} =$$

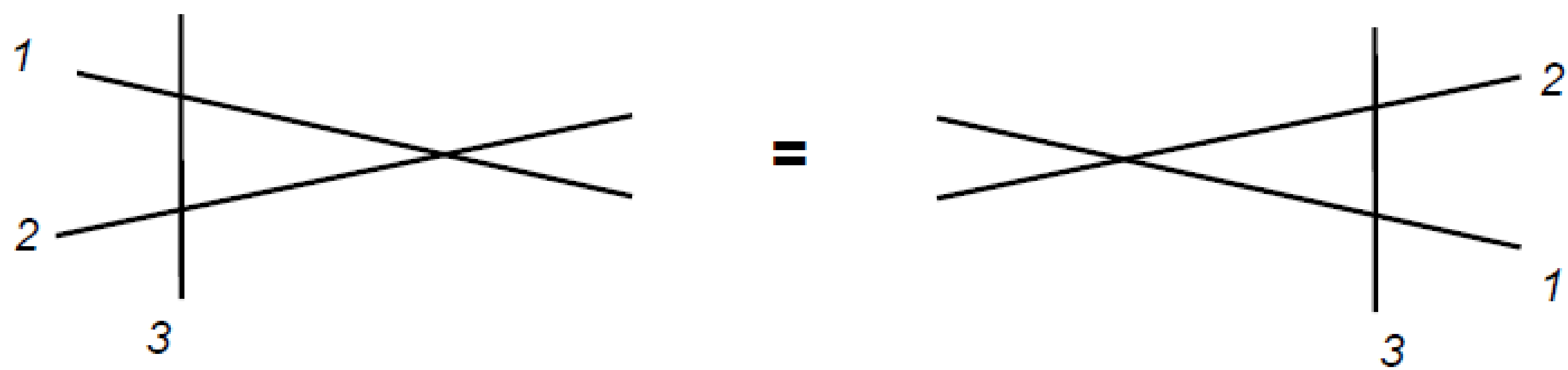


The Yang-Baxter relation

$$R(x-x')\,L_j(x)\otimes L_j(x') = L_j(x')\otimes L_j(x)\,R(x-x')$$

The R-matrix

$$R(x) = \begin{pmatrix} x+\eta & 0 & 0 & 0 \\ 0 & \eta & x & 0 \\ 0 & x & \eta & 0 \\ 0 & 0 & 0 & x+\eta \end{pmatrix}$$



Quantum transfer matrix

$$\begin{aligned} T(x) &= \mathrm{tr}\Big[L_1(x)L_2(x)\ldots L_N(x)\Big] \\ &= x^N + J_{N-1}x^{N-1} + \ldots + J_1 x + J_0 \end{aligned}$$

$[T(x), T(x')] = 0$ for any x, x' with the same η

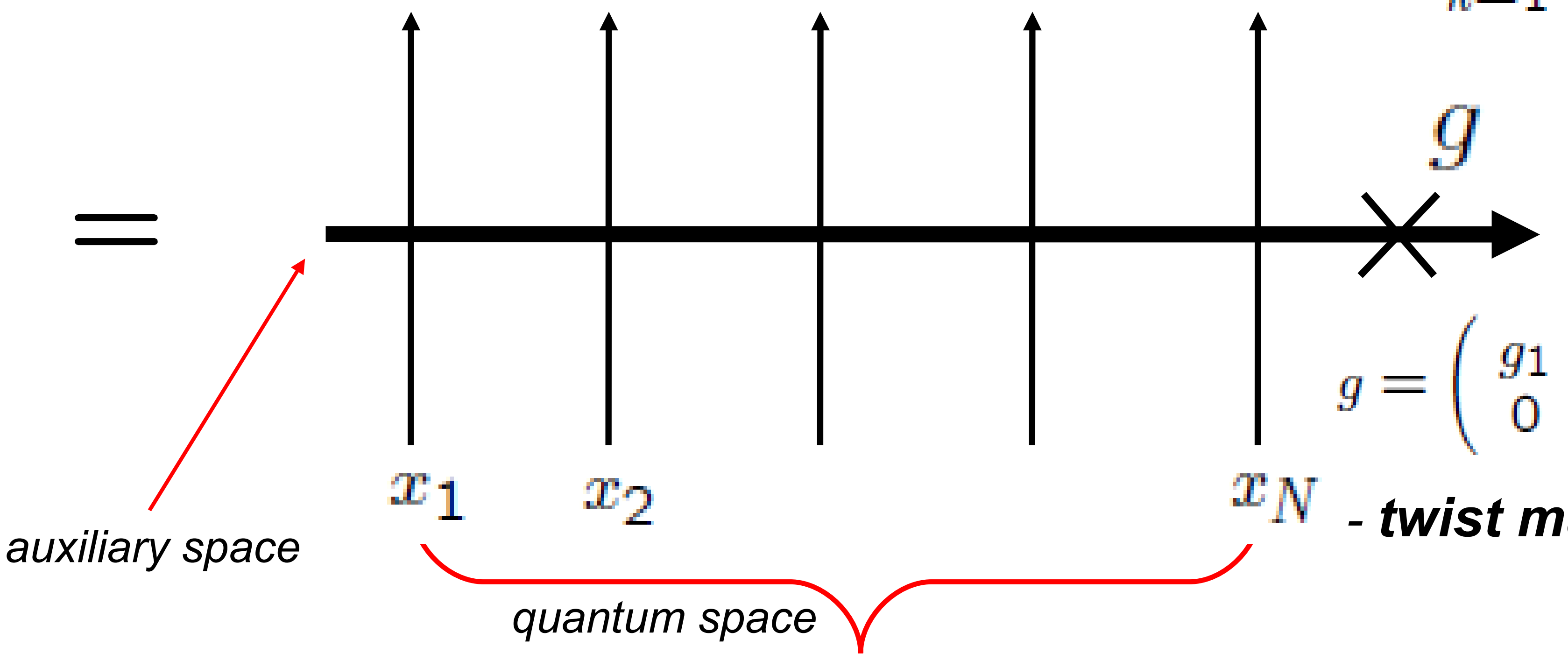
$$J_0 = T(0) = \eta^N P_{12}P_{23}P_{34}\ldots P_{N-1,N}P_{N,1}$$

$$H^{\mathrm{XXX}} = \eta \frac{d}{dx} \log T(x)\Big|_{x=0} - N$$

Generalization: inhomogeneous model with twisted boundary condition

$$T(x) = \mathrm{tr}\Big[L_1(x-x_1)L_2(x-x_2)\,\dots\,L_N(x-x_N)g\Big] = \sum_{k=1}^{N} H_k x^k$$

$=$



$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

- twist matrix

$[T(x), T(x')] = 0$ for any x, x' with the same η, g, x_i

The Hamiltonians are non-local

Another normalization:

$$\mathsf{T}(x) = \frac{T(x)}{\prod_{j=1}^{N}(x - x_j)} = \operatorname{tr} g + \sum_{j=1}^{N} \frac{H_i}{x - x_i}$$

Eigenvalues of T*(x) are given by*

$$\mathsf{T}(x) = p_1 \prod_{k=1}^{N} \frac{x - x_k + \eta}{x - x_k} \prod_{\alpha=1}^{M} \frac{x - u_\alpha - \frac{\eta}{2}}{x - u_\alpha + \frac{\eta}{2}} + p_2 \prod_{\alpha=1}^{M} \frac{x - u_\alpha + \frac{3\eta}{2}}{x - u_\alpha + \frac{\eta}{2}}$$

Bethe equations

$$\boxed{\frac{p_1}{p_2} \prod_{l=1}^{N} \frac{u_j - x_l + \frac{\eta}{2}}{u_j - x_l - \frac{\eta}{2}} = \prod_{k \neq j}^{M} \frac{u_j - u_k + \eta}{u_j - u_k - \eta}}$$

Limit to the Gaudin model $\eta \to 0$

$$g = e^{\eta h} \qquad h = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \quad g_1 = e^{\eta k_1},\; g_2 = e^{\eta k_2}$$

$$\mathrm{T}(x) = 2 + \eta\left(\operatorname{tr} h + \sum_i \frac{1}{x - x_i}\right) + \eta^2\left(\frac{1}{2}\operatorname{tr} h^2 + \sum_i \frac{\hat{H}_i^G}{x - x_i}\right) + O(\eta^3)$$

Gaudin Hamiltonians

$$\hat{H}_i^G = h^{(i)} + \sum_{k \neq i} \frac{P_{ij}}{x_i - x_j}$$

$$P_{ij} = \frac{1}{2}\left(1 + \vec{\sigma}^{(i)}\vec{\sigma}^{(j)}\right)$$

is the permutation operator

Eigenvalues of the Gaudin Hamiltonians:

$$H_i^G = k_1 + \sum_{j \neq i}^{N} \frac{1}{x_i - x_j} + \sum_{\alpha=1}^{M} \frac{1}{u_\alpha - x_i}$$

Bethe equations:

$$k_1 - k_2 + \sum_{j=1}^{N} \frac{1}{u_\alpha - x_j} = 2 \sum_{\beta \neq \alpha}^{M} \frac{1}{u_\alpha - u_\beta}$$

$$\alpha = 1, 2, \ldots, M$$

Calogero-Moser model

Exactly solvable many-body problem of <u>classical mechanics</u>

The Hamiltonian:

$$H = \sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{2}{(x_i - x_j)^2}$$

Equations of motion:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

$$\ddot{x}_i = -\sum_{j \neq i} \frac{8}{(x_i - x_j)^3}$$

Integrability of the Calogero-Moser model

Equations of motion can be written in the Lax form

$$\frac{d\mathsf{L}}{dt} = [\mathsf{A}, \mathsf{L}]$$

where

$$\mathsf{L}_{ij} = p_i \delta_{ij} + \frac{1-\delta_{ij}}{x_i - x_j}$$

(the Lax matrix)

$$\mathsf{A}_{ij} = -\delta_{ij} \sum_{k \neq i} \frac{2}{(x_i - x_j)^2} + \frac{2(1-\delta_{ij})}{(x_i - x_j)^2}$$

This means that

$$\mathsf{L}(t) = U(t)\mathsf{L}(0)U^{-1}(t)$$

and, therefore,

$$\frac{d}{dt} \det(z\mathsf{I} - \mathsf{L}(t)) = 0$$

The Lax matrix:

$$L = \begin{pmatrix} p_1 & \frac{1}{x_1 - x_2} & \frac{1}{x_1 - x_3} & \cdots & \frac{1}{x_1 - x_n} \\ \frac{1}{x_2 - x_1} & p_2 & \frac{1}{x_2 - x_3} & \cdots & \frac{1}{x_2 - x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n - x_1} & \frac{1}{x_n - x_2} & \frac{1}{x_n - x_3} & \cdots & p_n \end{pmatrix}$$

Hamiltonians in involution: $H_k = \mathrm{tr}\, \mathsf{L}^k$

In particular: $H_1 = -P\,, \qquad H_2 = H$

Explicit expression for integrals of motion:

$$\det(z\mathsf{I} - \mathsf{L}(t)) = \exp\left(\sum_{i<j} \frac{\partial_{p_i}\partial_{p_j}}{(x_i - x_j)^2}\right) \prod_{k=1}^{N} (z + p_k)$$

(K.Sawada and T.Kotera, 1975)

The quantum-classical (QC) duality

Quantum		*Classical*
Gaudin	⟷	**Calogero-Moser**

$$H_i^G = p_i$$

$$\mathsf{L} = \begin{pmatrix} p_1 & \frac{1}{x_1 - x_2} & \frac{1}{x_1 - x_3} & \cdots & \frac{1}{x_1 - x_n} \\ \frac{1}{x_2 - x_1} & p_2 & \frac{1}{x_2 - x_3} & \cdots & \frac{1}{x_2 - x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n - x_1} & \frac{1}{x_n - x_2} & \frac{1}{x_n - x_3} & \cdots & p_n \end{pmatrix}$$

Possible values of p_i ***such that spectrum of*** L ***is***

$$(\underbrace{k_1, \ldots, k_1}_{N-m}, \underbrace{k_2, \ldots, k_2}_{m})$$

correspond to eigenvalues of H_i^G

(a kind of <u>inverse spectral problem</u> for the Lax matrix)

Back to the inhomogeneous XXX chain

$$T(x) = \mathrm{tr}\Big[L_1(x-x_1)L_2(x-x_2)\ldots L_N(x-x_N)g\Big] = \sum_{k=1}^{N} H_k x^k$$

$$\mathsf{T}(x) = \frac{T(x)}{\prod_{j=1}^{N}(x-x_j)} = \mathrm{tr}\, g + \sum_{j=1}^{N}\frac{H_i}{x-x_i}$$

The spectrum of H_i is related, in a similar way, to possible

to possible values of particles velocities in the

classical <u>Ruijsenaars N-body model</u>

Ruijsenaars model

Exactly solvable many-body problem of <u>classical mechanics</u>

The Hamiltonian:

$$H = \sum_{j=1}^{N} e^{p_j} \prod_{k \neq j}^{N} \frac{x_j - x_k + 1}{x_j - x_k}$$

Equations of motion:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

$$\ddot{x}_i = -\sum_{k \neq i} \frac{2\,\dot{x}_i\,\dot{x}_k}{(x_i - x_k)((x_i - x_k)^2 - 1)}$$

Integrability of the Ruijsenaars model

Equations of motion can be written in the Lax form

$$\frac{d\mathsf{L}}{dt} = [\mathsf{A}, \mathsf{L}]$$

where $$L_{ij} = \frac{\dot{x}_i}{x_i - x_j + 1}$$ ***(the Lax matrix)***

The matrix A can be also written explicitly

This means that $$\mathsf{L}(t) = U(t)\mathsf{L}(0)U^{-1}(t)$$

and, therefore, $$\frac{d}{dt}\det(z\mathsf{I} - \mathsf{L}(t)) = 0$$

The Lax matrix

$$L = \begin{pmatrix} \dot{x}_1 & \frac{\dot{x}_1}{x_1 - x_2 + 1} & \frac{\dot{x}_1}{x_1 - x_3 + 1} & \cdots & \frac{\dot{x}_1}{x_1 - x_n + 1} \\ \frac{\dot{x}_2}{x_2 - x_1 + 1} & \dot{x}_2 & \frac{\dot{x}_2}{x_2 - x_3 + 1} & \cdots & \frac{\dot{x}_2}{x_2 - x_n + 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\dot{x}_n}{x_n - x_1 + 1} & \frac{\dot{x}_n}{x_n - x_2 + 1} & \frac{\dot{x}_n}{x_n - x_3 + 1} & \cdots & \dot{x}_n \end{pmatrix}$$

The quantum-classical (QC) duality

Quantum		*Classical*
XXX chain	⟷	**Ruijsenaars**

$$H_i = \dot{x}_i$$

$$\mathsf{L} = \begin{pmatrix} \dot{x}_1 & \frac{\dot{x}_1}{x_1-x_2+1} & \frac{\dot{x}_1}{x_1-x_3+1} & \cdots & \frac{\dot{x}_1}{x_1-x_n+1} \\ \frac{\dot{x}_2}{x_2-x_1+1} & \dot{x}_2 & \frac{\dot{x}_2}{x_2-x_3+1} & \cdots & \frac{\dot{x}_2}{x_2-x_n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\dot{x}_n}{x_n-x_1+1} & \frac{\dot{x}_n}{x_n-x_2+1} & \frac{\dot{x}_n}{x_n-x_3+1} & \cdots & \dot{x}_n \end{pmatrix}$$

Possible values of $\dot{x}_i$ *such that spectrum of* L *is*

$$(\underbrace{g_1, \dots, g_1}_{N-m}, \underbrace{g_2, \dots, g_2}_{m})$$

correspond to eigenvalues of H_i

(a kind of inverse spectral problem for the Lax matrix)