Parastatistics and Homotopy Algebras

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Nonlinear Mathematical Physics
and Natural Hazards
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joint work with Michel Dubois-Violette

Green Parastatistics

Creation-Annihilation Algebra of H.S.Green (1953)

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Basis-free definition of $\mathcal{F}$ generated in $V = \bigoplus_{i \in I} \mathbb{K} a_i^\dagger$

$$PS(V) = T(V)/([[V, V]_\otimes, V]_\otimes)$$

where $(\mathfrak{I})$ stands for a two-sided ideal generated by $\mathfrak{I}$

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The algebra $PS(V)$ is the Universal Enveloping Algebra of the graded Lie algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

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To be compared with $S(V) = U(V)$ *for nilpotent alg* $[V, V] = 0$

Parastatistics Algebra $PS(V)$ and Young Tableaux

Theorem

Let $S^{\lambda}(V)$ be the Schur module associated with Young diagram λ. The algebra $PS(V)$ is a $GL(V)$-model, i.e., every irreducible polynomial $GL(V)$-representations appears once and exactly once

$$PS(V) = \bigoplus_{\lambda} S^{\lambda}(V)$$

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Proof: The Cauchy formula is an identity of characters

$$\prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_\lambda s_\lambda(x)$$

where $s_\lambda(x) = chS^\lambda(V)$ stands for the Schur polynomial of dim V variables.

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$$+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_p \tag{5}$$

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Theorem (Chevalley-Eilenberg)

The chain complex $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$ is a resolution.

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Koszul complex of the Symmetric Algebra $S(V)$, $D = \dim V$

$$0 \to SV \otimes \Lambda^D V \to SV \otimes \Lambda^{D-1} V \to \cdots \tag{6}$$

$$\cdots \to S\,V \otimes \Lambda^2 V \to S\,V \otimes V \to S\,V \to \mathbb{K} \to 0 \tag{7}$$

Chevalley-Eilenberg for 2-nilpotent $\mathfrak{g} = V \oplus \Lambda^2 V$

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The minimal resolution reads $\mathbf{P}_\bullet = PS \otimes H_\bullet(\mathfrak{g}, \mathbb{K})$

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$$\mathbf{P}: \qquad 0 \to P_d \to \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{K}$$

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The elements of $H^\bullet(\mathfrak{g},\mathbb{K}) \overset{1-1}{\longleftrightarrow}$ self-conjugated Young Tableaux!

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sum over all trees with $n = r + s + t$ leafs.

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Theorem (Kadeishvili)

Let A be a DGA and let $H^\bullet(A)$ be the cohomology ring of A. There is an A_∞- algebra structure on $H^\bullet(A)$ with $m_1 = 0$ and m_2 induced by the multiplication on A, constructed from the DGA A, such that there is a quasi-isomorphism of A_∞-algebras $H^\bullet(A) \xrightarrow{\iota} A$ lifting the identity of $H^\bullet(A)$.

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$\mu\epsilon\tau\alpha\phi o\rho\alpha \equiv$ *transport, transfer'*

Homotopy Metaphor Theorem

Cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ as a C_∞ algebra

Lie algebras (finite) and commutative DG algebras

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Proof.

Apply the Kadeishvili Homotopy transfer theorem to the commutative DG algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

$$(\Lambda^p \mathfrak{g}^*, \delta^p) \quad \text{and} \quad H^\bullet(\Lambda^p \mathfrak{g}^*, \delta^p)$$

Via a metric g, one gets identified $\mathfrak{g}^* \stackrel{g}{\cong} \mathfrak{g}$, $\delta^p = \partial^*_{p+1}$, $h_p = \partial_p$

$$\iota\pi - Id_{\Lambda \mathfrak{g}^*} = \partial\partial^* + \partial^*\partial =: \Delta \qquad \ker\Delta = H^\bullet(\mathfrak{g}, \mathbb{K})$$

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$H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ *is generated in degree 1 as C_∞-algebra.*

$$\boxed{\alpha_i} \in V^* = H^1(\mathfrak{g}, \mathbb{K})$$

$$m_2(\boxed{\alpha_1}, \boxed{\alpha_2}) = \pi(\boxed{\alpha_1} \wedge \boxed{\alpha_2}) = \pi\left(\begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \end{array}\right) = 0$$

$$m_3(\boxed{\alpha_1}, \boxed{\alpha_2}, \boxed{\alpha_3}) = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \in H^2(\mathfrak{g}, \mathbb{K})$$

$$m_3\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \boxed{\alpha_4}, \boxed{\alpha_5}\right) = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_4 & \multicolumn{2}{c}{} \\ \cline{1-1} \alpha_5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \in H^3(\mathfrak{g}, \mathbb{K})$$

$$m_2\left(\begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \boxed{\alpha_4}\right) = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \alpha_4 \\ \hline \end{array} \qquad \in H^3(\mathfrak{g}, \mathbb{K})$$

REFERENCE

M. Dubois-Violette, T. Popov. Young tableaux and homotopy commutative algebra. arXiv:1202.2230

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Thank you for your attention!