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Operadic Bridge between Renormalization Theory and Vertex Algebras

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The aim of this talk is to show how two different areas in QFT are governed by one and the same algebraic structure. Then I will discuss the perspectives of transfering constructions in both directions via this common structure.

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Renormalization Group and its action in perturbative QFT

There is an operad, which we call

Expansion operad $\mathcal{E}$

whose algebras are the vertex (or OPE) algebras

the group associated to this operad is the renorm. group

- What is a vertex algebra ?
- What is an operad ?
- What is the renorm. group and its action ?

  (representation by formal diffeomorpisms on the physical parameters)

A vertex algebra is the structure closed by the OPE. The OPE (Wilson, 1964) was introduced for the analysis of the short distance behaviour in QFT.

$$\phi(x)\,\psi(y) \underset{x\to y}{\sim} \sum_{A} \Theta_A(y)\, C_A(x-y)$$

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- In higher D: Borcherds (1998), N. (2005) for QFT with G.C.I. (N, Todorov 2001)

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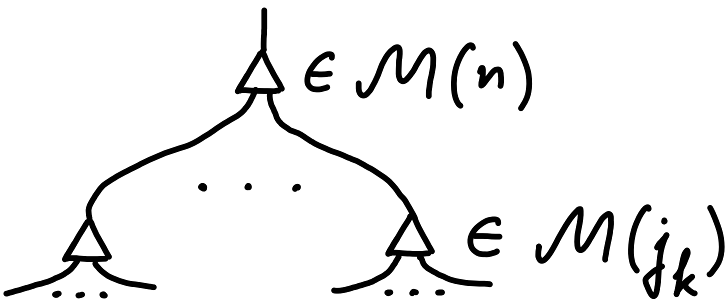

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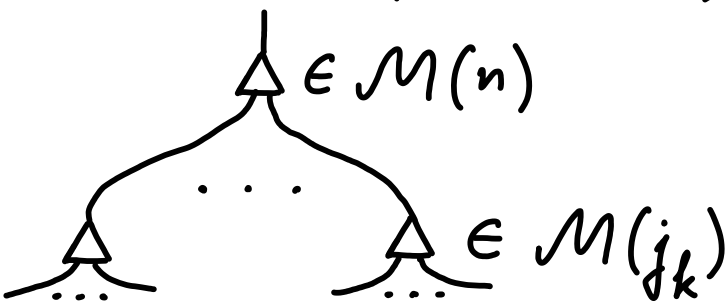

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- Representation $\equiv$ algebra over an operad

$\{\mathcal{M}(n)\}_n \to \{End_V(n)\}_n$ - morphism

i.e., $\mathcal{M}(n) \to Hom(V^{\otimes n}, V)$

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$$= \sum_\ell G'_\ell(x_j,\dots,x_{j+k})\, G''_\ell(x_1,\dots,x_{j-1},x_{j+k},\dots,x_n)$$

for $|x_a - x_{j+k}| \ll |x_b - x_{j+k}|$

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In the applications to vertex algebras and renormalization theory for massless fields:

$\mathcal{O}_n$ = The algebra of rational n-point functions on $\mathbb{R}^D \ni x_1, \ldots, x_n$

$$\frac{P(x_1, \ldots, x_n)}{\prod\limits_{j<k} \left((x_j - x_k)^2\right)^{\nu_{j,k}}}$$ with "light-cone" singularities

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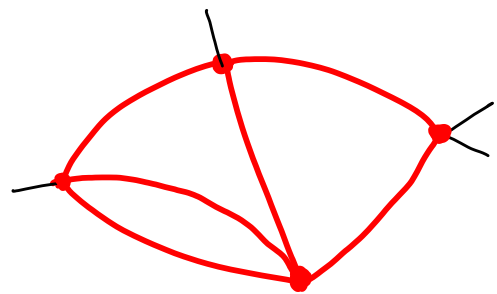

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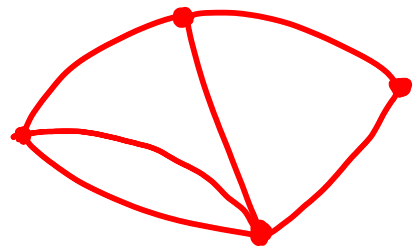

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Ultraviolet renormalizations on configuration space in terms of Feynman amplitudes:

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... Bogolubov ... Epstein-Glaser ...

Recent: N. arXiv 2009; N., Stora, Todorov 2013

Renormalization ambiguity at order n:

$$\left\{ Q : \mathcal{O}_n \to \mathcal{D}'[0] \;\middle|\; \begin{array}{l}\text{commuting with}\\ \text{multiplication by}\\ \text{polynomials}\end{array} \right\}$$

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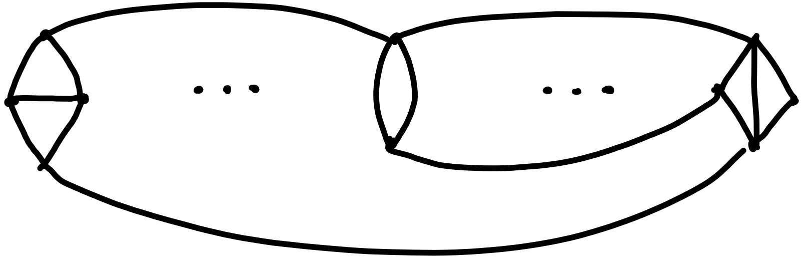
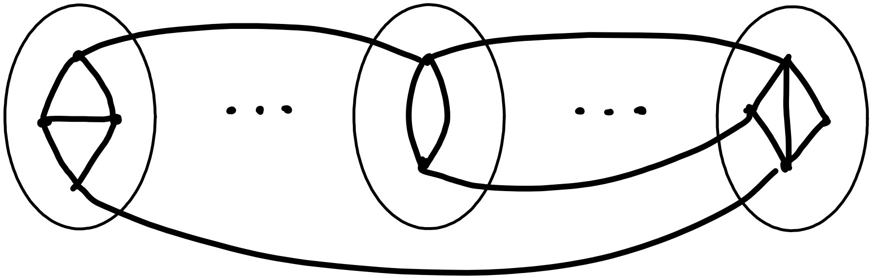
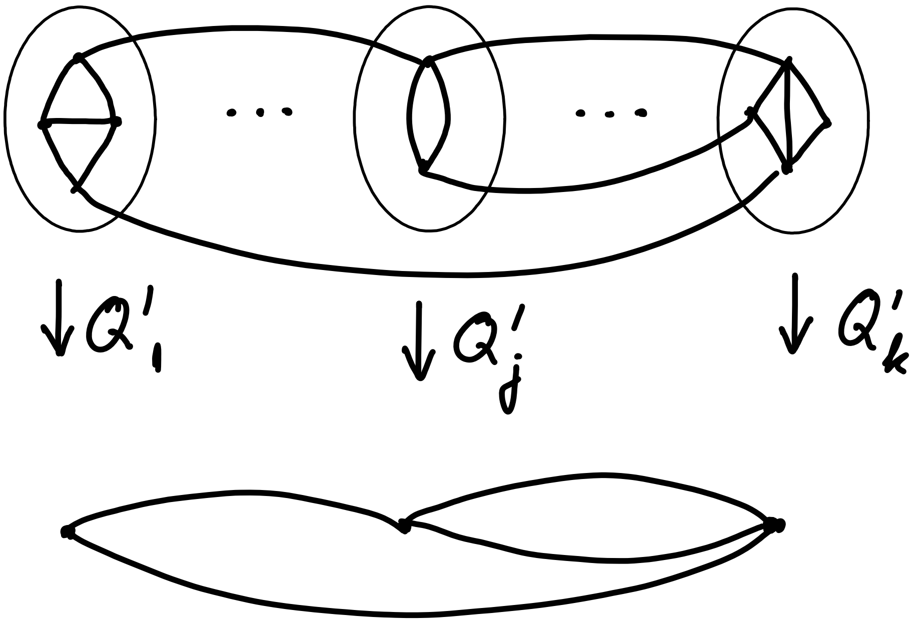
Furthermore, the operadic compositions in $\mathcal{E}(n)$ have an interpretation on $\mathcal{R}(n)$ that corresponds to basic operations used in the renormalization group composition.



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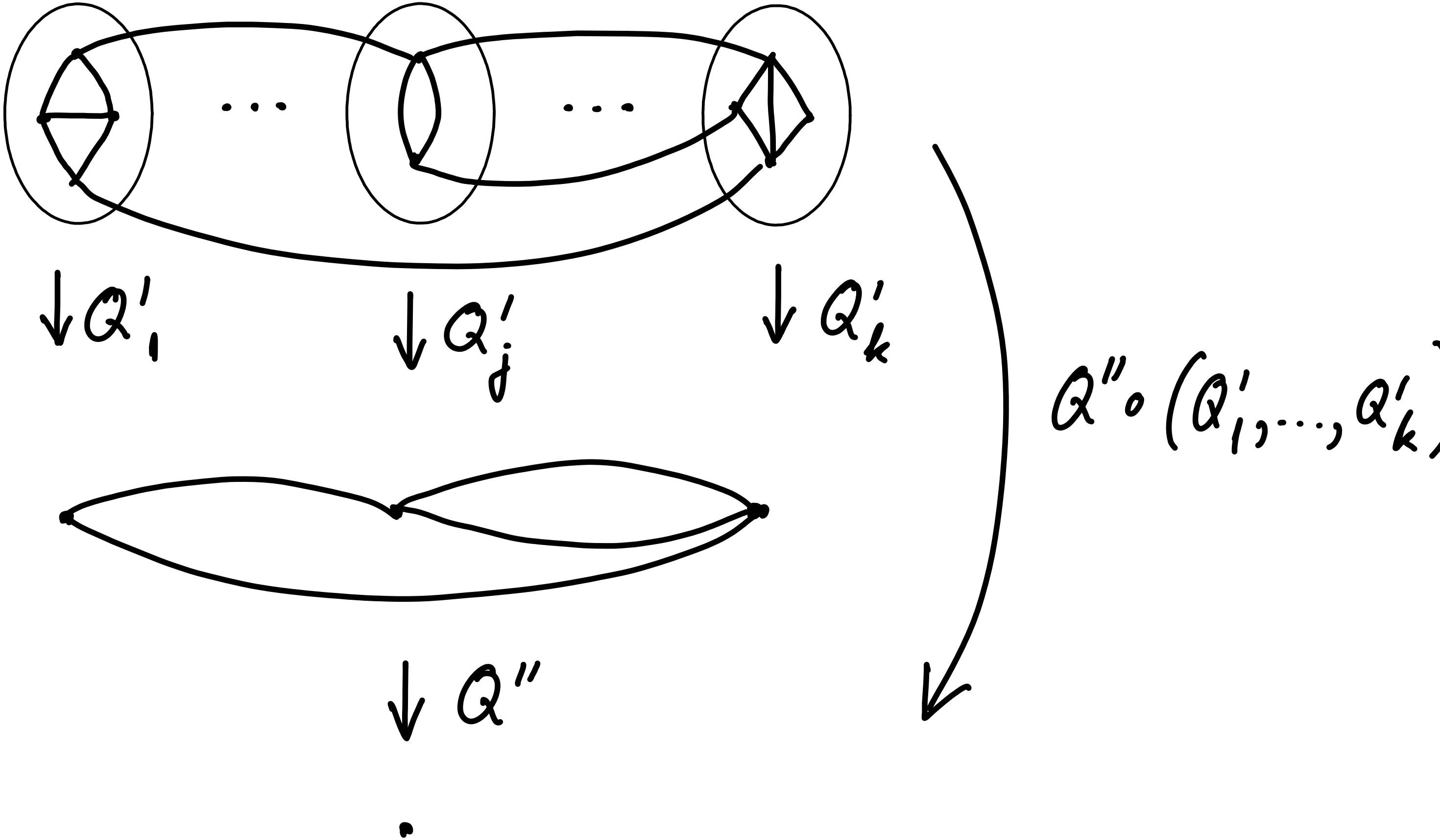
Q'_j
Q'_k



Q'_1
Q'_j
Q'_k
$Q'' \circ (Q'_1, \dots, Q'_k)$
Q''

Stückelberg-Bogolubov renormalization group: the ambiguity in the recursive renormalization at order n

$$
\begin{tikzcd}
\{ \mathcal{O}_n \longrightarrow \mathcal{D}' \} \arrow[d, "\{Q'_n\}"] \arrow[dd, bend left=60, "\{Q'''_n \in \mathcal{E}(n)\}"] \\
\{ \mathcal{O}_n \longrightarrow \mathcal{D}' \} \arrow[d, "\{Q''_n\}"] \\
\{ \mathcal{O}_n \longrightarrow \mathcal{D}' \}
\end{tikzcd}
$$

$\boxed{\text{Loday}}$, N. 2011

We obtain a functor

$$\{\text{Operads}\} \longrightarrow \{\text{Groups}\}$$

whic produces:

- Renorm. group when applied to $\mathcal{E}$
- The group of formal diff. when appl. on End_V
- Renorm. group action via $\mathcal{E} \to End_V$.

The $\mathcal{E}$-algebras are vertex algebras

$$\uparrow$$

Expansion operad $\mathcal{E}$

$$\downarrow$$

The group associated to $\mathcal{E}$ is the renorm. group