Chaotic vs Regular Behavior in Yang–Mills Theories

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Gauge invariance dictates dynamics

Abelian Theory

$$\Psi \longrightarrow e^{-ix}\,\Psi$$

$$A_\mu \longrightarrow A_\mu - \frac{1}{e}\partial_\mu x$$

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

Non – Abelian Theory (Yang-Mills)

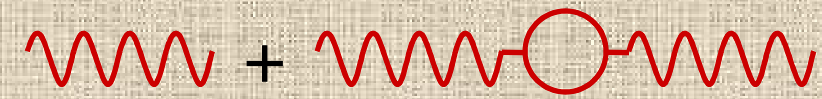
$$\Psi \longrightarrow U\,\Psi$$

$$\vec{A}_\mu \longrightarrow \frac{i}{e}(\partial_\mu U)U^{-1} + U\,\vec{A}_\mu\,U^{-1}$$

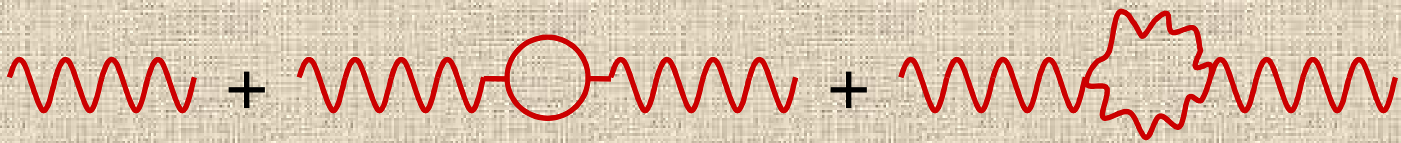
$$\vec{F}_{\mu\nu} = \partial_\nu\,\vec{A}_\mu - \partial_\mu\,\vec{A}_\nu + e\,\vec{A}_\mu \times \vec{A}_\nu$$

- Abelian gauge bosons are neutral (photon)
- Non-Abelian gauge bosons carry charges (gluons)

Quantum Loop Corrections



photon



gluon

Loop corrections are absorbed into the coupling constants
(running coupling constants)

QED $\alpha\,(Q^2)$ rises as Q^2 increases

QCD $\alpha_s(Q^2)$ decreases as Q^2 rises

QCD at large Q^2 – short distances:

The coupling constant is small.
Use perturbation (jet structure, scaling violations).

QCD at small Q^2 – large distances:

The coupling is large. Use non-perturbative techniques, or meaningful approximations.

An approximation

The low-energy, long wavelength limit of QCD, relevant for the ground state of QCD.

For large λ, the Yang-Mills fields are homogeneous in space, and they depend only upon time.

For a SU(2) pure Yang-Mills system

$$L = -\frac{1}{4g^2} F^{\alpha}_{\mu\nu} F^{\alpha}_{\mu\nu}$$ where

$$F^{\alpha}_{\mu\nu} = \partial_{\mu} A^{\alpha}_{\nu} - \partial_{\nu} A^{\alpha}_{\mu} + \varepsilon^{abc} A^{b}_{\mu} A^{c}_{\nu}$$

The gluon fields depend only on time:

$$A^{\alpha}_{i} = A^{\alpha}_{i}(t)$$

We select the gauge $A_0^\alpha = 0$

The classical equations of motion become

$$\ddot{A}_i^\alpha + (A_i^a A_j^b - A_j^a A_i^b) A_j^b = 0$$

We adopt the ansatz $A_i^\alpha = O_i^a f^a(t)$

where $O_i^a O_i^b = \delta^{ab}$

With $f^1 = x,\ f^2 = y,\ f^3 = z,$

the equations of motion are reproduced by the Hamiltonian

$$H = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}[x^2 y^2 + z^2 x^2 + y^2 z^2]$$

For the simplified case $z = 0$, the ‘particle’ is under the influence of the potential

$$V(x, y) = \frac{1}{2} x^2 y^2$$

Motion bounded by the hyperbola $x\,y = \pm(2E)^{1/2}$

Abelian solution

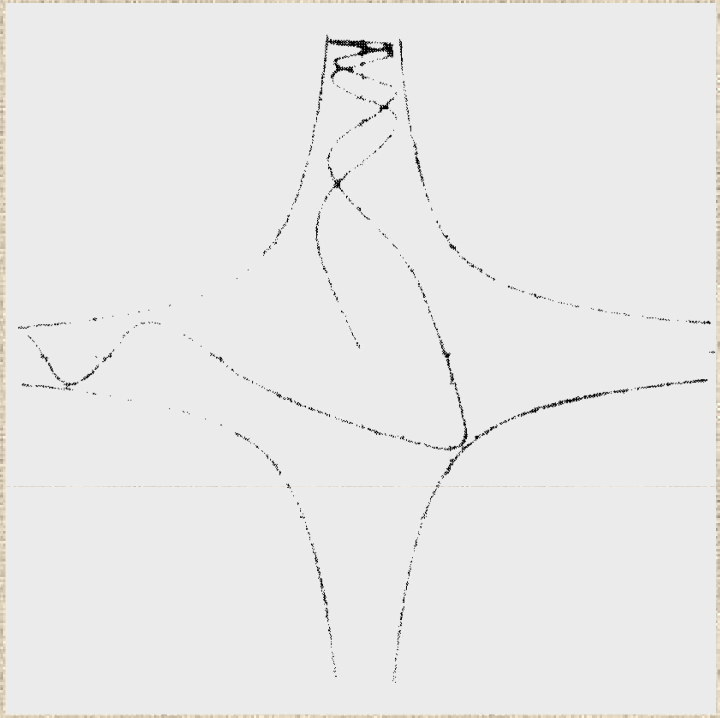
$\ddot{x}=0$ and $\dot{x}\neq 0$, or $\ddot{y}=0$ and $\dot{y}\neq 0$

(escape to infinity)

In general, a particle moving in one of the four ‘channels’ will return after a finite time to the central region $x \approx \mathrm{y}$.

There, in random fashion, the ‘particle’ chooses another color direction.

Sinai hyperbolic billiard



Special interest: the symmetric solution $x = y = J$ (Jacobi elliptic cosine). The solution is unstable. Overall <u>the system is chaotic</u>.

Include Quantum Corrections

A different ground state?

(color confinement, gluon condensation, chiral symmetry breaking).

Replace the fixed coupling g

by a running coupling

$$\bar{g}^2(\mu) = \frac{1}{b \ln\left(\frac{\mu^2 + \sigma^2}{\Lambda^2}\right)}$$

Quantum corrections generate logarithms of the chromomagnetic field. Identify the scale μ with the chromomagnetic field.

$$\bar{g}^2 = \frac{1}{b \ln\left(\dfrac{x^2 y^2 + \sigma^2}{\Lambda^2}\right)}$$

The effective Lagrangian becomes

$$L_{eff} = \frac{b}{2} \ln\left(\frac{x^2 y^2 + \sigma^2}{\Lambda^2} \right) [\dot{x}^2 + \dot{y}^2 - \dot{x}^2 \dot{y}^2]$$

and the Hamiltonian

$$H_{eff} = (p_x^2 + p_y^2) \Big/ (2b \ln u) + \frac{b}{2} x^2 y^2 \ln u$$

with $u = \dfrac{\sigma^2 + x^2 y^2}{\Lambda^2}$

and $p_x = b\dot{x} \ln u, \; p_y = b\dot{y} \ln u$

The motion is bounded by

$$x\,y = c(E)$$

where c is defined by

$$c^2 \ln\left(\frac{\sigma^2 + c^2}{\Lambda^2}\right) = 2\,\frac{E}{b}$$

Poincaré Section

$$H = E = const. \quad y = 0 \quad p_y > 0$$

In the Poincaré map $p_x{}^2$ is bounded by

$$p_x^2 < 2\,E\,b\,\ln\left(\frac{\sigma^2}{\Lambda^2}\right)$$

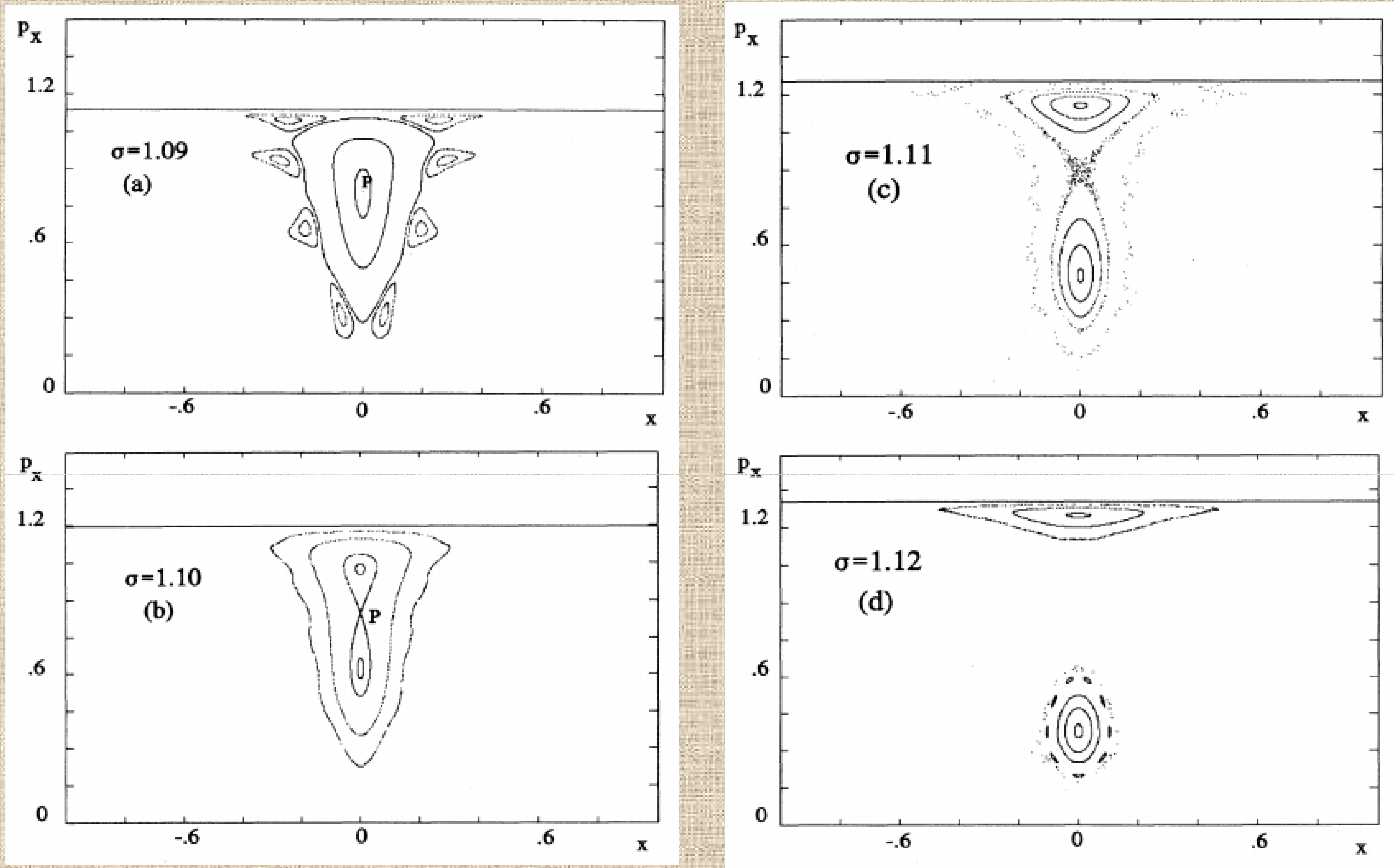
We revisit the symmetric

periodic solution $x = y$

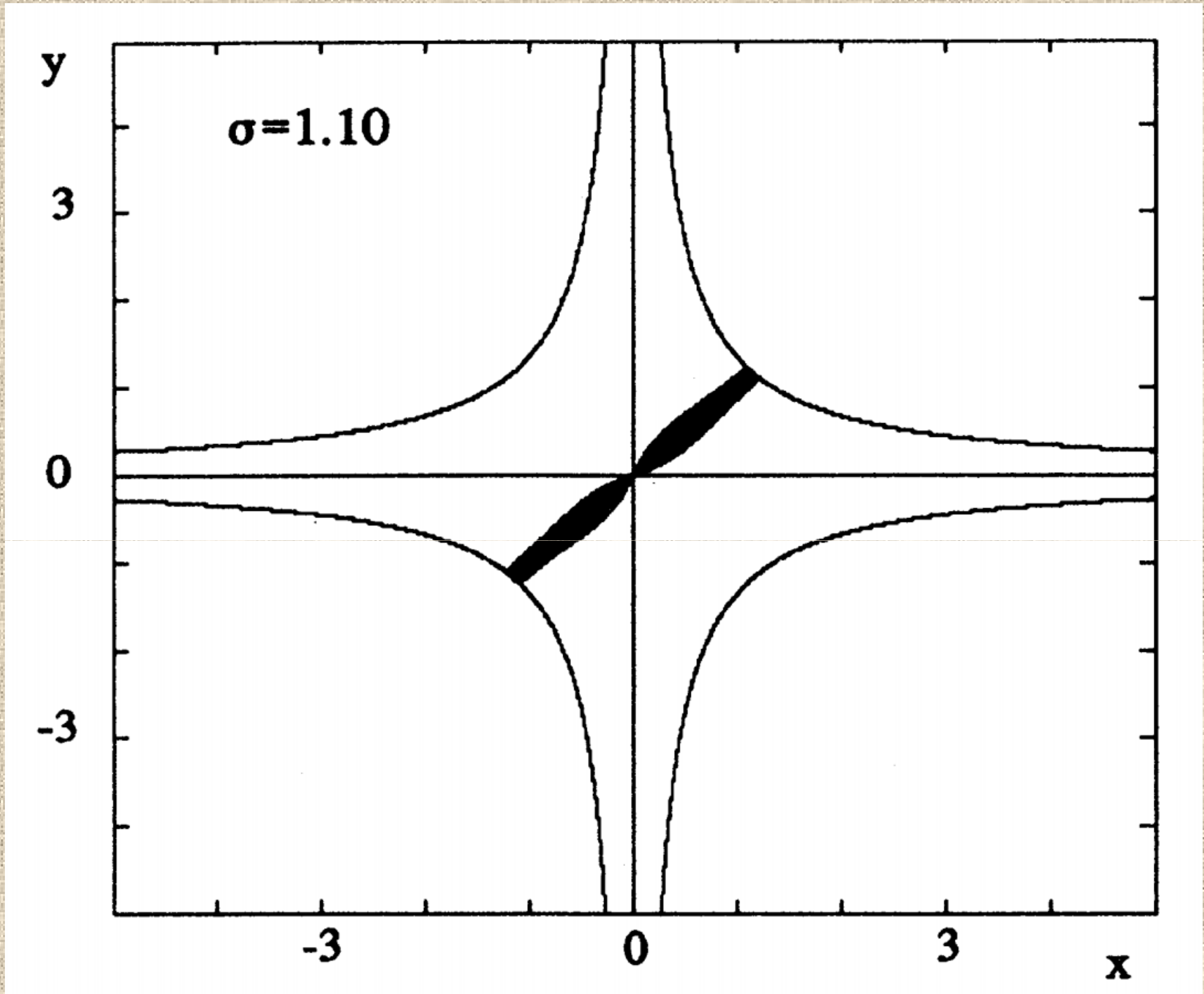
Previously it was always unstable.

The quantum corrections stabilize it for

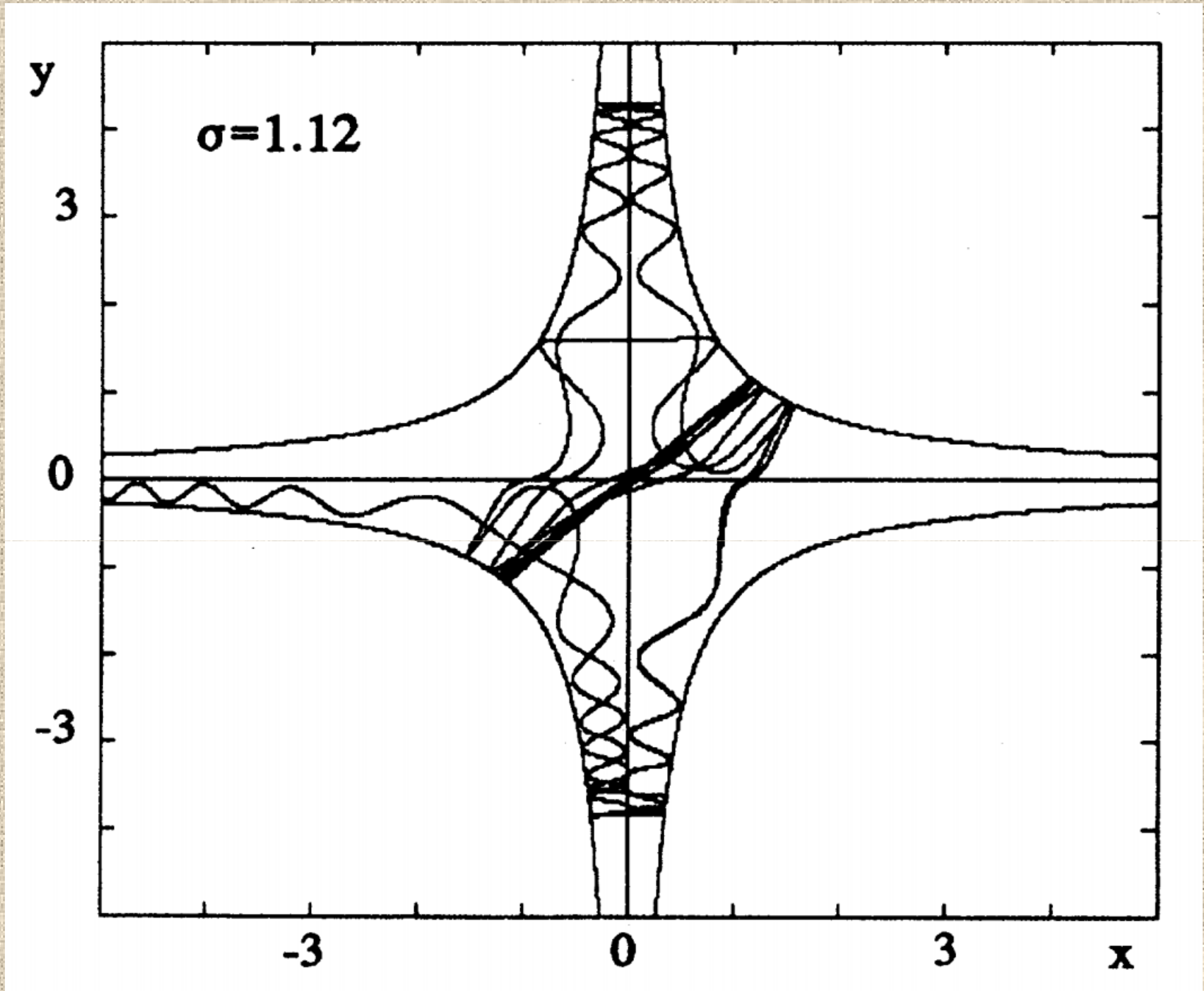
values of σ, Λ, E in suitable open domains.

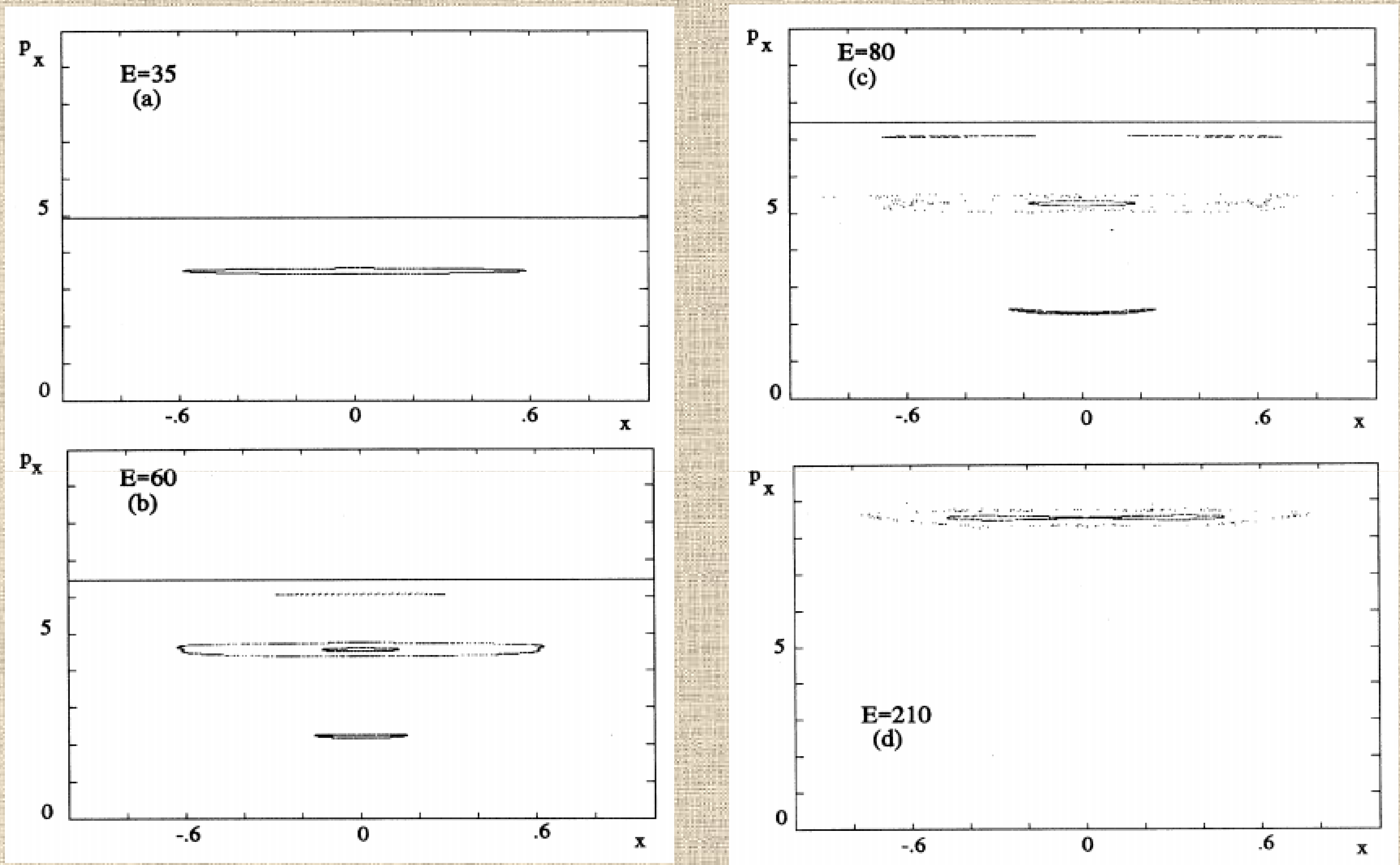


p_x
1.2
.6
0
-.6
0
.6
x
σ=1.09
(a)
P
σ=1.10
(b)
P
σ=1.11
(c)
σ=1.12
(d)



y
σ=1.10
3
0
-3
-3
0
3
x


y
σ=1.12
3
0
-3
-3
0
3
x



p_x
E=35
(a)
5
0
-.6
0
.6
x
p_x
E=60
(b)
5
0
-.6
0
.6
x
p_x
E=80
(c)
5
0
-.6
0
.6
x
p_x
5
E=210
(d)
0
-.6
0
.6
x

Overall

The classical Hamiltonian displays chaotic behavior.

The quantum corrections introduce new scales. The symmetric solution $x = y$, which is unstable at the classical level, becomes stabilized at the quantum level.

A stable symmetric solution exists also in the full 3-dimensional problem. The solution $x = y = z$ represents a color-neutral gluonic field, a sort of a glueball.