# Bianchi cosmological models as integrable geodesic flows

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- Homogeneous cosmological models
- Geodesic flow on symmetric spaces

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Forty two years after the famous paper of

**Francesco Calogero**

*Solution of the one-dimensional $n$-body problem with quadratic and/or inversely quadratic pair potentials*

Journal of Mathematical Physics 12 (1971) 419-436

**Bill Sutherland**

Physical Review A 1971, 1972

**Jürgen Moser**

Adv. Math. 1975

The **Calogero-Sutherland-Moser** type system, which consists of $n$-particles on a line interacting with pairwise potential $V(x)$ admits the following spin generalization

$$H_{ECM} = \frac{1}{2}\sum_{a=1}^{n} p_a^2 + \frac{1}{2}\sum_{a\neq b}^{n} f_{ab} f_{ba} V(x_a - x_b)$$

where

| | | |
|---|---|---|
| **I.** | $\mathbf{V(z)} = z^{-2}$ | **Calogero** |
| **II.** | $\mathbf{V(z)} = a^2 \sinh^{-2} az$ | |
| **III.** | $\mathbf{V(z)} = a^2 \sin^{-2} az$ | **Sutherland** |
| **IV.** | $\mathbf{V(z)} = a^2 \wp(az)$ | |
| **V.** | $\mathbf{V(z)} = z^{-2} + \omega z^2$ | **Calogero** |

The nonvanishing Poisson bracket relations

$$\{x_i, p_j\} = \delta_{ij}\,, \qquad \{f_{ab}, f_{cd}\} = \delta_{bc} f_{ad} - \delta_{ad} f_{cb}$$

The **Calogero-Sutherland-Moser** type system, which consists of $n$-particles on a line interacting with pairwise potential $V(x)$ admits the following spin generalization

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The configuration space $\mathcal{S}$ of the real symmetric $3 \times 3$ matrices can be endowed with the flat Riemannian metric

$$ds^2 =< dQ\,, dQ >= \mathbf{Tr}\, dQ^2\,,$$

whose group of isometry is formed by orthogonal transformations $Q' = R^T\, Q\, R$. The system is invariant under the orthogonal transformations $S' = R^T\, S\, R$. The orbit space is given as a quotient space $\mathcal{S}/SO(3, \mathbb{R})$ which is a stratified manifold

(1) *Principal orbit-type stratum,*
when all eigenvalues are unequal $x_1 < x_2 < x_3$
with the smallest isotropy group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$

(2) *Singular orbit-type strata*
forming the boundaries of orbit space with
  (a) two coinciding eigenvalues (e.g. $x_1 = x_2$) ,
  when the isotropy group is $SO(2) \otimes \mathbb{Z}_2$
  (b) all three eigenvalues are equal $(x_1 = x_2 = x_3)$ ,
  here the isotropy group coincides with the isometry group $SO(3,\mathbb{R})$

**Let us consider the Hamiltonian system with the phase space spanned by the $N \times N$ symmetric matrices $X$ and $P$ with the noncanonical symplectic form**

$$\{X_{ab}, P_{cd}\} = \frac{1}{2}\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) .$$

**The Hamiltonian of the system defined as**

$$H = \frac{1}{2}\mathbf{tr}P^2$$

**describes a free motion in the matrix configuration space.**

**The following statement is fulfilled:**

*The Hamiltonian rewritten in special coordinates coincides with the Euler-Calogero-Moser Hamiltonian*

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2}$$

**with nonvanishing Poisson brackets for the canonical variables**

$$\{x_i, p_j\} = \delta_{ij} \,,$$
$$\{l_{ab}, l_{cd}\} = \frac{1}{2}\left(\delta_{ac}l_{bd} - \delta_{ad}l_{bc} + \delta_{bd}l_{ac} - \delta_{bc}l_{ad}\right) \,.$$

**Let us introduce new variables**

$$X = O^{-1}(\theta)Q(x)O(\theta)\,,$$

**where the orthogonal matrix** $O(\theta)$ **is parameterized by the** $\frac{N(N-1)}{2}$ **variables, e.g., the Euler angles** $(\theta_1, \cdots, \theta_{\frac{N(N-1)}{2}})$ **and** $Q = diag\|x_1, \cdots, x_N\|$ **denotes a diagonal matrix. This point**

transformation induces the canonical one which we can obtain using the generating function

$$F_4 = \left[P, x_1, \cdots, x_N, \theta_1, \cdots, \theta_{\frac{N(N-1)}{2}}\right] = \mathbf{tr}[X(x,\theta)P].$$

Using the representation

$$P = O^{-1}\left[\sum_{a=1}^{N} \bar{\alpha}_a \bar{P}_a + \sum_{i<j=1}^{\frac{N(N-1)}{2}} \alpha_{ij} P_{ij}\right] O,$$

where the matrices $(\bar{\alpha}_a, \alpha_{ij})$ form an orthogonal basis in the space of the symmetric $N \times N$ matrices under the scalar product

$$\mathbf{tr}(\bar{\alpha}_a\bar{\alpha}_b) = \delta_{ab}\,, \qquad \mathbf{tr}(\alpha_{ij}\alpha_{kl}) = 2\delta_{ik}\delta_{jl}\,, \qquad \mathbf{tr}(\alpha_a\alpha_{ij}) = 0\,,$$

one can find that $\bar{P}_a = p_a$ and components $P_{ab}$ are represented via the $O(N)$ right invariant vectors fields $l_{ab}$

$$P_{ab} = \frac{1}{2}\frac{l_{ab}}{x_a - x_b}\,.$$

## The integration of the Hamilton equations of motion

$$\dot{X} = P\,, \qquad \dot{P} = 0$$

**derived with the help of Hamiltonian gives the solution of the Euler-Calogero-Moser Hamiltonian system as follows:** *for the $x$-coordinates we need to compute the eigenvalues of the matrix $X = X(0) + P(0)t$, while the orthogonal matrix $O$, which diagonalizes $X$, determines the time evolution of internal variables.*

# Forty six years after the famous papers of

Morikazu Toda

*Journal of Physical Society of Japan 20 (1967) 431*

*Journal of Physical Society of Japan 20 (1967) 2095*

The Hamiltonian of the non-periodic Toda lattice is

$$H_{NPT} = \frac{1}{2}\sum_{a=1}^{n} p_a^2 + g^2 \sum_{a=1}^{n-1} \exp\left[2(x_a - x_{a+1})\right],$$

and the Poisson bracket relations are

$$\{x_i, p_j\} = \delta_{ij}\,.$$

The equations of motion for the non-periodic Toda lattice are

$$\begin{aligned} \dot{x}_a &= p_a,\ a = 1, 2, \ldots, n\,, \\ \dot{p}_a &= -2\ \exp\left[2(x_a - x_{a+1})\right] + 2\ \exp\left[2(x_{a-1} - x_a)\right], \\ \dot{p}_1 &= -2\ \exp\left[2(x_1 - x_2)\right], \\ \dot{p}_n &= -2\ \exp\left[2(x_{n-1} - x_n)\right], \end{aligned}$$

and for the periodic Toda lattice

$$\begin{aligned} \dot{x}_a &= p_a\,, \\ \dot{p}_a &= 2\ \exp\left[2(x_{a-1} - x_a)\right] - \exp\left[2(x_a - x_{a+1})\right]. \end{aligned}$$

Further for simplicity we put $k_{ab} = 0$ and $\alpha = 0$.

We use the Gauss decomposition for the positive-definite symmetric matrix $S$

$$S = Z\,D\,Z^T\,.$$

Here $D = \mathbf{diag}\,\|x_1, x_2, \dots x_n\|$ is a diagonal matrix with positive elements and $Z$ is an upper triangular matrix

$$Z = \begin{pmatrix} 1 & z_{12} & \dots & z_{1n} \\ 0 & 1 & \dots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

The differential of the symmetric matrix $S$ is given by

$$dS = Z\left[dD + D\Omega + (D\Omega)^T\right]Z^T\,,$$

where $\Omega$ is a matrix-valued right-invariant 1-form defined by

$$\Omega := dZZ^{-1}\,.$$

In the Lie algebra $gl(n,\mathbb{R})$ of $n\times n$ real matrices we introduce Weyl basis with elements

$$(e_{ab})_{ij} = \delta_{ai}\,\delta_{bj}\,,$$

**which are $n \times n$ matrices in the form**

$$e_{ab} = \begin{pmatrix} & & \vdots & \\ \dots & \dots & \dots \; 1 & \dots \\ & & \vdots & \\ & & \vdots & \\ & & \vdots & \end{pmatrix}.$$

**The scalar product in $gl(n,\mathbb{R})$ is defined by**

$$(e_{ab}\,, e_{cd}) = \mathbf{tr}(e_{ab}^T\,, e_{cd}) = \delta_{ac}\,\delta_{bd}\,.$$

Let us introduce also the matrices

$$\bar{\alpha}_a = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

**and**

$$\alpha_{ab} = \begin{pmatrix} & \vdots & & \vdots & \\ \dots & \dots & \dots & 1 & \dots \\ & \vdots & & \vdots & \\ \dots & 1 & \dots & \vdots & \dots \\ & \vdots & & \vdots & \end{pmatrix}.$$

with scalar product is defined by

$$
\begin{aligned}
(\bar\alpha_a\,,\bar\alpha_b) &= \mathbf{tr}(\bar\alpha_a\,\bar\alpha_b) = \delta_{ab}\,,\\
(\alpha_{ab}\,,\alpha_{cd}) &= \mathbf{tr}(\alpha_{ab}\,\alpha_{cd}) = 2\,(\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd})\ ,\\
(\bar\alpha_a\,,\alpha_{bc}) &= \mathbf{tr}(\bar\alpha_a\,\alpha_{bc}) = 0\,.
\end{aligned}
$$

Now we can write the differential of the matrix $S$ in the form

$$dS = Z \left[ \sum_{a=1}^{n} dx_a \bar{\alpha}_a + \sum_{a<b=1}^{n} x_a \Omega_{ab}\, \alpha_{ab} \right] Z^T \,.$$

Here $\Omega_{ab}$ are the coefficients of the matrix $\Omega$

$$\Omega = \sum_{a<b} \Omega_{ab}\, e_{ab} \,.$$

In the case of Gauss decomposition of the symmetric matrix $S = ZDZ^T$ the corresponding momenta we seek in the form

$$P = (Z^T)^{-1}\left(\sum_{a=1}^{n} \bar{\mathcal{P}}_a \bar{\alpha}_a + \sum_{a<b}^{n} \mathcal{P}_{ab}\,\alpha_{ab}\right) Z^{-1}\,.$$

From the condition of invariance of the symplectic 1-form

$$\mathbf{tr}\,(PdS) = \sum_{i=1}^{n} p_a\, dx_a + \sum_{a<b=1}^{n} p_{ab}\, dz_{ab}\,,$$

where the new canonical variables $(x_a\,, p_a)\,, (z_{ab}\,, p_{ab})$ obey the Poisson bracket relations

$$\{x_a\,, p_b\} = \delta_{ab}\,, \qquad \{z_{ab}\,, p_{cd}\} = \delta_{ac}\,\delta_{bd}\,,$$

we obtain

$$\bar{\mathcal{P}}_a = p_a\,, \qquad \mathcal{P}_{ab} = \frac{l_{ab}}{2x_a}\,.$$

Here $l_{ab}$ are right-invariant vector fields on the group of upper triangular matrices with unities on the diagonal

$$l_{ab} = \left(\Omega^{-1}\right)^{T}_{ab,cd} \, p_{cd} \, ,$$

where $\Omega_{ab,cd}$ are coefficients in the decompositions of the 1-forms $\Omega_{ab}$ in coordinate basis

$$\Omega_{ab} = \Omega_{ab,cd} \, dz_{cd} \, ,$$

The canonical Hamiltonian in the new variables takes the form

$$H = \frac{1}{2}\sum_{a=1}^{n} p_a^2\, x_a^2 + \frac{1}{4}\sum_{a<b=1}^{n} l_{ab}^2\, \frac{x_a}{x_b}\,.$$

After the canonical transformation

$$x_a = e^{y_a}\,, \qquad p_a = \pi_a\, e^{-y_a}$$

we arrive to Hamiltonian in the form of
**spin nonperiodic Toda chain**

$$H = \frac{1}{2}\sum_{a=1}^{n}\pi_a^2 + \frac{1}{4}\sum_{a<b=1}^{n} l_{ab}^2\, e^{y_a - y_b}\,.$$

The generators of the group of upper triangular matrices are

$$l_{12} = p_{12} + \sum_{k=2}^{n} z_{2k}\, p_{1k}\,, \qquad l_{13} = p_{13} + \sum_{k=2}^{n} z_{2k}\, p_{1k}\,.$$

The internal variables satisfy the Poison bracket relations

$$\{l_{ab}\,, l_{cd}\} = C^{ef}{}_{ab,cd}\, l_{ef}$$

with structure coefficients

$$C^{ef}{}_{ab,cd} = \delta_{bc}\,\delta_{ae}\,\delta_{df}\,.$$

**The generalized Toda chains are integrable systems associated with Lie algebras.**

$$H = \frac{1}{2}\sum_{a=1}^{n} \pi_a^2 + \sum_{k=1}^{r} l_k^2 \, e^{2\, q_{\alpha_k}} \,,$$

**where** $q_{\alpha_k} =< \alpha_k, q >$.

# Type $A_{n-1}$

## The simple roots

$$\alpha_1 = e_1 - e_2\,, \ldots\,, \alpha_{n-1} = e_{n-1} - e_n\,.$$

## The potential

$$U = e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n}\,.$$

## Type $B_n$

### The simple roots

$$\alpha_1 = e_1 - e_2\,, \ldots\,, \alpha_{n-1} = e_{n-1} - e_n\,, \alpha_n = e_n\,.$$

### The potential

$$U = e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{q_n}\,.$$

# Type $C_n$

## The simple roots

$$\alpha_1 = e_1 - e_2\,, \ldots\,, \alpha_{n-1} = e_{n-1} - e_n\,, \alpha_n = 2e_n\,.$$

## The potential

$$U = e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{2q_n}\,.$$

## Type $D_n$

### The simple roots

$$\alpha_1 = e_1 - e_2\,, \ldots\,, \alpha_{n-1} = e_{n-1} - e_n\,, \alpha_n = e_{n-1} + e_n\,.$$

### The potential

$$U = e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{q_{n-1} + q_n}\,.$$

## Type $G_2$

### The simple roots

$$\alpha_1 = e_1 - e_2\,, \alpha_2 = -2e_1 + e_2 + e_3\,.$$

### The potential

$$U = e^{q_1 - q_2} + e^{-2q_1 + q_2 + q_3}\,.$$

# Type $F_4$

## The simple roots

$$\alpha_1 = e_1 - e_2\,, \alpha_2 = e_2 - e_3\,, \alpha_3 = e_3\,, \alpha_4 = 1/2\,(e_4 - e_1 - e_2 - e_3)\ .$$

## The potential

$$U = e^{q_1 - q_2} + e^{\frac{1}{2}(q_4 - q_1 - q_2 - q_3)}\,.$$

# Type $E_6$

## The simple roots

$$\alpha_1 = 1/2\,(-e_1 + e_2 + \cdots + e_7 - e_8)\,, \alpha_2 = e_1 - e_2\,,$$
$$\alpha_3 = e_2 - e_3\,, \alpha_4 = e_3 - e_4\,, \alpha_5 = e_4 - e_5\,, \alpha_6 = -\,(e_1 + e_2)\ .$$

## The potential

$$U = e^{q_1 - q_2} + e^{q_2 - q_3} + e^{q_3 - q_4} + e^{q_4 - q_5} + e^{-(q_1 + q_2)} + \\ + e^{\frac{1}{2}(-q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7 - q_8)}\,.$$

# Homogeneous cosmological models

## Spacetime decomposition

Let $(\mathcal{M}, g)$ be a smooth four-dimensional paracompact and Hausdorf manifold. In each point of the open set $\mathcal{U} \subset \mathcal{M}$ we propose that are defined the local basis of 1-forms and its dual basis

$$\{\omega^\mu,\ \mu\ = 0,\ 1,\ 2,\ 3\} \qquad \{e_\nu,\ \nu\ = 0,\ 1,\ 2,\ 3\}$$

such that

$$[e_\alpha, e_\beta] = C^\mu{}_{\alpha\beta} e_\mu,$$

where $C^\mu{}_{\alpha\beta}$ are the basis structure functions. The symmetric metric is

$$g = g_{\mu\nu}\omega^{\mu} \otimes \omega^{\nu}\,, \qquad g^{-1} = g^{\mu\nu} e_{\mu} \otimes e_{\nu}$$

We suppose that on the manifold $(\mathcal{M}, g)$ is defined affine connection $\nabla$

$$\Gamma^{\mu}{}_{\nu} = \Gamma^{\mu}{}_{\nu\alpha}\omega^{\alpha} \Longleftrightarrow \Gamma^{\mu}{}_{\beta\alpha} = \langle \omega^{\mu}, \nabla_{e_{\alpha}} e_{\beta} \rangle$$

In manifold with affine connection we can construct the bilinear antisymmetric mapping

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X\,,Y]$$

The First Cartan structure equation is

$$d\omega^\mu + \Gamma^\mu{}_\alpha \wedge \omega^\alpha = \frac{1}{2} T^\mu$$

The tensor type $(1,3)$ defined by

$$R(\omega,\; Z,\; X,\; Y) = \langle \omega,\; R(\;X,\; Y) Z \rangle$$

is called the curvature tensor, where

$$R(\;X,\; Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [\nabla_X, \nabla_Y]$$

If we define the curvature 2-form $\Omega^{\mu}{}_{\nu}$ by

$$\Omega^{\mu}{}_{\nu} = d\Gamma^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\alpha} \wedge \Gamma^{\alpha}{}_{\nu}$$

one can obtain the Second structure Cartan equation

$$\Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta}\, \omega^{\alpha} \wedge \omega^{\beta}$$

In the Riemannian geometry the commutator of two vector fields is given by

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

and for every vector field $X$ we have the condition $\nabla_X g = 0$.

The connection and the Riemann tensor in the noncoordinate basis $\{e_\nu\}$ are given with

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left( e_\alpha g_{\beta\sigma} + e_\beta g_{\alpha\sigma} - e_\sigma g_{\alpha\beta} \right)$$
$$- \frac{1}{2} g^{\mu\sigma} (g_{\alpha\rho} C^{\rho}{}_{\beta\sigma} + g_{\beta\rho} C^{\rho}{}_{\alpha\sigma}) - \frac{1}{2} C^{\mu}{}_{\alpha\beta} \,,$$
$$R^{\sigma}{}_{\alpha\mu\nu} = e_\mu \Gamma^{\sigma}{}_{\alpha\nu} - e_\nu \Gamma^{\sigma}{}_{\alpha\mu} + \Gamma^{\rho}{}_{\alpha\nu} \Gamma^{\sigma}{}_{\rho\mu} - \Gamma^{\rho}{}_{\alpha\mu} \Gamma^{\sigma}{}_{\rho\nu} - \Gamma^{\sigma}{}_{\alpha\rho} C^{\rho}{}_{\mu\nu} \,,$$

where $C^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\beta\alpha} - \Gamma^{\mu}{}_{\alpha\beta}$.

Hereinafter we suppose that

$$\mathcal{M} = T_1 \times \Sigma_t$$

Let us introduce the noncoordinate basis of vector fields $\{e_\perp, e_a\}$

$$[e_\perp, e_a] = C^\perp{}_{\perp a} e_\perp + C^d{}_{\perp a} e_d \,,$$
$$[e_a, e_b] = C^d{}_{ab} e_d \,,$$

with the structure functions

$$C^\perp{}_{\perp a} = e_a ln N, \qquad C^d{}_{\perp a} = \frac{1}{N}(N^b C^d{}_{ab} + e_a N^d)$$

The metric in the corresponding dual basis $\{\theta^\perp, \theta^a\}$

$$g = -\theta^\perp \otimes \theta^\perp + \gamma_{ab}\, \theta^a \otimes \theta^b,$$

where $\gamma$ is the induced metric on the submanifold $\Sigma_t$. The components of the vector field $X_0$ in this basis

$$X_0 = Ne_\perp + N^a e_a$$

are the Lagrange multipliers in ADM scheme which can be obtained if we pass to the coordinate basis $\{X_0 = \frac{\partial}{\partial t}, \frac{\partial}{\partial x^a}\}$

$$g = -\left(N^2 - N^a N_a\right) dt \otimes dt + 2N_a dt \otimes dx^a + \gamma_{ab} dx^a \otimes dx^b$$

The second fundamental form characterizes the embeding of

## $\Sigma_t$ in $(\mathcal{M}, g)$

$$K(X,Y) : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$$

$$K(X,Y) = \frac{1}{2}(Y.\nabla_X e_\perp - X.\nabla_Y e_\perp), \qquad X, Y \in T\Sigma_t$$

**The another representation for the extrinsic curvature is**

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{e_\perp}\gamma_{ab}$$

**To find the $3+1$ decomposition we define the induced on**

## the $\Sigma_t$ connection

$$ {}^3\Gamma^c{}_{ba} := \Gamma^c{}_{ba} = \langle {}^3\theta^c, {}^3\nabla_{e_a} e_b \rangle, $$

where ${}^3\nabla_X$ is the covariant derivative with respect to $\gamma$ and corresponding curvature tensor is

$$ {}^3R(X,Y) = {}^3\nabla_X {}^3\nabla_Y - {}^3\nabla_Y {}^3\nabla_X - [{}^3\nabla_X, {}^3\nabla_Y] $$

In the basis $\{e_\perp, e_a\}$ we find the componets of the connection

$$\begin{aligned}
\Gamma^{\perp}{}_{\perp\perp} &= 0, & \Gamma^{\perp}{}_{\perp i} &= 0, \\
\Gamma^{j}{}_{\perp\perp} &= c^{j}, & \Gamma^{j}{}_{\perp i} &= -K^{j}{}_{i}, \\
\Gamma^{\perp}{}_{i\perp} &= c_{i}, & \Gamma^{\perp}{}_{ji} &= -K_{ji}, \\
\Gamma^{j}{}_{i\perp} &= -K^{j}{}_{i} + \langle {}^{3}\theta^{j}, \mathcal{L}_{e_{\perp}} e_{i} \rangle, & \Gamma^{k}{}_{ij} &= {}^{3}\Gamma^{k}{}_{ij}
\end{aligned}$$

**and of the Riemann tensor**

$$R^{j}{}_{\perp k \perp} = K^{js} K_{sk} + \gamma^{js} \mathcal{L}_{e_{\perp}} K_{ks} + \frac{1}{N} \gamma^{js\,3} \nabla_{e_k} {}^{3}\nabla_{e_s} N \,,$$

$$R^{\perp}{}_{ijk} = {}^{3}\nabla_{e_k} K_{ij} - {}^{3}\nabla_{e_j} K_{ik} \,,$$

$$R^{s}{}_{ijk} = {}^{3}R^{s}{}_{ijk} + K_{ik} K_{j}{}^{s} - K_{ij} K_{k}{}^{s}$$

Finally the scalar curvature can be obtained in the form

$$R = {}^3R + K_a{}^a K_b{}^b - 3K_{ab}K^{ab} - \frac{2}{N}\gamma^{ab3}\nabla_{e_a}{}^3\nabla_{e_b}N - 2\gamma^{ab}\mathcal{L}_{e_\perp}K_{ab}$$

The Hilbert-Einstein action is

$$L[N, N_a, \gamma_{ab}, \dot{N}, \dot{N}_a, \dot{\gamma}_{ab}] = \iint_{t_1 S}^{t_2} dt\ {}^3\sigma N\left\{{}^3R + K_a{}^a K_b{}^b - 3K_{ab}K^{ab}\right\}$$

$$- \iint_{t_1 S}^{t_2} dt\ {}^3\sigma N\left\{\frac{2}{N}\gamma^{ab}\ {}^3\nabla_{e_a}\ {}^3\nabla_{e_b}N - 2\gamma^{ab}\ \mathcal{L}_{e_\perp}K_{ab}\right\}$$

where ${}^3\sigma = \sqrt{\gamma}\ \theta^1 \wedge \theta^2 \wedge \theta^3$ is the volume 3-form on $\Sigma_t$.

# Bianchi models

**L. Bianchi 1897**

*Sugli spazii atre dimensioni du ammettono un gruppo continuo di movimenti*

**Soc. Ital. della Sci. Mem. di Mat.**

**(Dei. XL) (3) 3 267**

**By definition, Bianchi models are manifolds with product topology**

$$\mathcal{M} = \mathbb{R} \times G_3$$

**On the three dimensional Riemannnian manifold $\Sigma_t \gamma$ there**

exist left-invariant 1-forms $\{\chi^a\}$ such that

$$d\chi^a = -\frac{1}{2} C^a{}_{bc}\, \chi^b \wedge \chi^c$$

The dual vector fields $\{\xi_a\}$ form a basis in the Lie algebra of the group $G_3$

$$[\xi_a, \xi_b] = C^d{}_{ab} \xi_d$$

with structure constants $C^d{}_{ab} = 2\, d\chi^d(e_a, e_b)$. In the invariant basis

$$[e_\perp, e_a] = C^d{}_{\perp a} e_d, \qquad [e_a, e_b] = -C^d{}_{ab} e_d,$$

with $C^d{}_{\perp a} = N^{-1} N^b C^d{}_{ab}$ the metric takes the form

$$g = -\,\theta^\perp \otimes \theta^\perp + \gamma_{ab}\, \theta^a \otimes \theta^b$$

The preferable role of this choice for a coframe is that the functions $N, N^a$ and $\gamma_{ab}$ depend on the time parameter $t$ only. Due to this simplification the initial variational problem for Bianchi A models is restricted to a variational problem of the "mechanical" system

$$L\left(N, N_a, \gamma_{ab}, \dot{\gamma}_{ab}\right) = \int_{t_1}^{t_2} dt\, \sqrt{\gamma} N \left[\, {}^3R - K_a{}^a K_b{}^b + K_{ab} K^{ab} \,\right],$$

where ${}^3R$ is the curvature scalar formed from the spatial metric $\gamma$

$${}^3R = -\frac{1}{2}\gamma^{ab} C^c{}_{da} C^d{}_{cb} - \frac{1}{4}\gamma^{ab}\gamma^{cd}\gamma_{ij} C^i{}_{ac} C^j{}_{bd},$$

Table 1

**The Bianchi-Behr classification of groups**

| class | type | N | | | a |
|---|---|---|---|---|---|
| A | I | 0 | 0 | 0 | 0 |
| | II | 1 | 0 | 0 | 0 |
| | $VI_0$ | 0 | 1 | -1 | 0 |
| | $VII_0$ | 0 | 1 | 1 | 0 |
| | VIII | 1 | 1 | -1 | 0 |
| | IX | 1 | 1 | 1 | 0 |

Table 2

**The Bianchi-Behr classification of groups**

| | | | | | |
|---|---|---|---|---|---|
| B | V | 0 | 0 | 0 | 1 |
| | IV | 0 | 0 | 1 | 1 |
| | $\mathrm{VI}_a$, $a < 0$ | 0 | 1 | -1 | $\sqrt{-a}$ |
| | $(\mathrm{III}=\mathrm{VI}_{-1})$ | | | | |
| | $\mathrm{VII}_a$, $a > 0$ | 0 | 1 | 1 | $\sqrt{a}$ |

**and**

$$K_{ab} = -\frac{1}{2N}\left( (\gamma_{ad}C^d{}_{bc} + \gamma_{bd}C^d{}_{ac})N^c + \dot{\gamma}_{ab} \right)$$

Table 3

**Relation between Thurston's geometries and BKS types**

| Thurston's geometries | BKS types | class | sectional curvature |
|---|---|---|---|
| $\mathbb{R}^3$ | I, $\mathrm{VII}_0$ | A | $0, 0, 0$ |
| $\mathbb{S}^3$ | IX | A | $1, 1, 1$ |
| $\mathbb{H}^3$ | V | B | $-1, -1, -1$ |
| | $\mathrm{VII}_a,\ a > 0$ | | $-a^2, -a^2, -a^2$ |
| $\mathbb{S}^2 \times \mathbb{R}$ | K-S | | $1, 0, 0$ |

Table 4

**Relation between Thurston's geometries and BKS types**

| | | | |
|---|---|---|---|
| $\widetilde{SL2\mathbb{R}}$ | VIII | A | $-\frac{5}{4}, -\frac{1}{4}, -\frac{1}{4}$ |
| $Nil$ | II | A | $-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ |
| $Sol$ | $\mathrm{VI}_0$ | A | $1, -1, -1$ |

is the extrinsic curvature of the slice $\Sigma_t$ defined by the relation

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{e_\perp}\gamma_{ab}$$

In the theory we have four primary constraints

$$\pi := \frac{\delta L}{\delta \dot{N}} = 0\,, \qquad \pi^a := \frac{\delta L}{\delta \dot{N}_a} = 0$$

$$\pi^{ab} := \frac{\delta L}{\delta \dot{\gamma}_{ab}} = \sqrt{\gamma}(\gamma_{ab}K^i{}_i - K^{ab})$$

The symplectic structure on the phase space is defined by the following non-vanishing Poisson brackets

$$\{N, \pi\} = 1, \qquad \{N_a, \pi^b\} = \delta_a^b, \qquad \{\gamma_{cd}, \pi^{rs}\} = \frac{1}{2}\left(\delta_c^r \delta_d^s + \delta_c^s \delta_d^r\right)$$

Due to the reparameterization symmetry of inherited from the diffeomorphism invariance of the initial Hilbert-Einstein action, the evolution of the system is unambiguous and it is governed by the total Hamiltonian

$$H_T = N\mathcal{H} + N^a \mathcal{H}_a + u_0 P^0 + u_a P^a$$

with four arbitrary functions $u_a(t)$ and $u_0(t)$. One can verify that the secondary constraints are first class and obey the algebra

$$\{\mathcal{H}, \mathcal{H}_b\} = 0, \quad \{\mathcal{H}_a, \mathcal{H}_b\} = -C^d{}_{ab}\mathcal{H}_d$$

From the condition of time conservation of the primary constaraints folllows four secondary constaraints

$$\mathcal{H} = \frac{1}{\sqrt{\gamma}}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^a{}_a\pi^b{}_b\right) - \sqrt{\gamma}\ {}^3R, \qquad \mathcal{H}_a = 2\ C^d{}_{ab}\pi^{bc}\gamma_{cd},$$

which obey the algebra

$$\{\mathcal{H}, \mathcal{H}_a\} = 0, \qquad \{\mathcal{H}_a, \mathcal{H}_b\} = -C^d{}_{ab}\mathcal{H}_d$$

The Hamiltonian form of the action for the Bianchi A models can be obtained in the form

$$L[N, N_a, \gamma_{ab}, \pi, \pi^a, \pi^{ab}] = \int_{t_1}^{t_2} \pi^{ab} d\gamma_{ab} - H_C dt$$

where the canonical Hamiltonian is a linear combination of the secondary constraints $H_C = N\mathcal{H} + N^a\mathcal{H}_a$.

# Hamiltonian reduction of Bianchi I cosmology

In the case of Bianchi I model the group which acts on $(\Sigma_t, \gamma)$ is $\mathbf{T}^3$ and the action takes the form

$$L[N, N_a, \gamma_{ab}, \pi, \pi^a, \pi^{ab}] = \int_{t_1}^{t_2} \pi^{ab} d\gamma_{ab} - N\mathcal{H}dt\,.$$

Using the decomposition for arbitrary symmetric non-singular matrix

$$\gamma = R^T(\chi) e^{2X} R(\chi)\,,$$

**where $X = diag\|x_1, x_2, x_3\|$ is diagonal matrix and**

$$R(\psi, \theta, \phi) = e^{\psi J_3} e^{\theta J_1} e^{\phi J_3}$$

**we can pass to the new canonical variables**

$$(\gamma_{ab}, \pi^{ab}) \Longrightarrow (\chi_a, p_{chi_a}; x_a, p_a)$$

**The corresponding momenta are**

$$\pi = R^T \left( \sum_{s=1}^{3} \overline{\mathcal{P}}_s \, \overline{\alpha}_s + \sum_{s=1}^{3} \mathcal{P}_s \, \alpha_s \right) R$$

where

$$\overline{\mathcal{P}}_a = p_a \, ,$$
$$\mathcal{P}_a = -\frac{1}{4}\frac{\xi_a}{\sinh(x_b - x_c)}, \quad (cyclic \;\; permutations \;\; a \neq b \neq c)$$

and the left-invariant basis of the action of the $SO(3, \mathbb{R})$ in the phase space with three dimensional orbits is given by

$$\xi_1 = \frac{\sin\psi}{\sin\theta}\; p_\phi + \cos\psi\; p_\theta - \sin\psi\cot\theta\; p_\psi,$$
$$\xi_2 = -\frac{\cos\psi}{\sin\theta}\; p_\phi + \sin\psi\; p_\theta + \cos\psi\cot\theta\; p_\psi \;\; ,$$
$$\xi_3 = p_\psi$$

In the new variables the Hamiltonian constraint reads

$$\mathcal{H}_{BI} = \frac{1}{2}\sum_{a=1}^{3} p_a^2 - \sum_{a<b} p_a p_b + \frac{1}{2}\sum_{(abc)} \frac{\xi_c^2}{\sinh^2(x_a - x_b)}$$

Another form for the Bianchi I Hamiltonian is

$$\mathcal{H}_{BI} = \frac{1}{2}\sum_{a=1}^{3} p_a^2 - \sum_{a<b} p_a p_b + \frac{1}{4}\sum_{a<b=1}^{3} l_{ab}^2\, e^{y_a - y_b}\,.$$

# Hamiltonian reduction of Bianchi II cosmology

By definition on the three dimensional spacelike submanifold acts the Heisenberg group $H(1)$. The right-invariant vector fields we choose to satisfy the Poisson bracket relations

$$\left\{\xi_1^R, \xi_2^R\right\} = \xi_3^R\,,$$
$$\left\{\xi_1^R, \xi_2^R\right\} = 0\,,$$
$$\left\{\xi_1^R, \xi_2^R\right\} = 0\,.$$

If we use the Gauss decomposition for the 3-metric

$$\gamma = Z^T \, D \, Z$$

we obtain the Hamiltonian constraint for the Bianchi II model in the form

$$\mathcal{H}_{BII} = p_1 + p_2 + p_3 - \frac{1}{2} \left(p_1 + p_2 + p_3\right) + \frac{1}{2} \, l_{23}^2 \, e^{y_3 - y_2} + \frac{1}{2} \, e^{2y_1} \, .$$

Many thanks

TO THE ORGANIZERS

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