The application of the discriminantly separable polynomials in the dynamical systems

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Kowalevski top [S. Kowalevski Acta Math. (1889)]

$$I_1 = I_2 = 2I_3,\ I_3 = 1$$

$$c = Mgx_0,\ y_0 = 0,\ z_0 = 0$$

The equations of motion:

$$\begin{aligned} 2\dot{p} &= qr \\ 2\dot{q} &= -pr - c\gamma_3 \\ \dot{r} &= c\gamma_2 \\ \dot{\gamma}_1 &= r\gamma_2 - q\gamma_3 \\ \dot{\gamma}_2 &= p\gamma_3 - r\gamma_1 \\ \dot{\gamma}_3 &= q\gamma_1 - p\gamma_2. \end{aligned} \tag{1}$$

Change of variables:

$$x_i = p \pm \imath q,\ i = 1,2$$
$$e_i = x_i^2 + c(\gamma_1 \pm \imath\gamma_2),\ i = 1,2. \tag{2}$$

The first integrals:

$$r^2 = E + e_1 + e_2$$
$$rc\gamma_3 = F - x_2 e_1 - x_1 e_2$$
$$c^2\gamma_3^2 = G + x_2^2 e_1 + x_1^2 e_2 \tag{3}$$
$$e_1 e_2 = k^2,$$

with

$$E = 6l_1 - (x_1 + x_2)^2,\ F = 2cl + x_1 x_2 (x_1 + x_2),\ G = c^2 - k^2 - x_1^2 x_2^2$$

Transformation of the first integrals

$$e_1 R(x_2) + e_2 R(x_1) + R_1(x_1, x_2) + k^2(x_1 - x_2)^2 = 0$$

with

$$\begin{aligned} R(x_i) &= x_i^2 E + 2x_i F + G \\ &= -x_i^4 + 6l_1 x_i^2 + 4lcx_i + c^2 - k^2, \quad i = 1, 2 \\ R_1(x_1, x_2) &= EG - F^2 \\ &= -6l_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4lc(x_1 + x_2)x_1 x_2 \\ &+ 6l_1(c^2 - k^2) - 4l^2 c^2. \end{aligned}$$

Kowalevski denotes

$$R(x_1, x_2) = Ex_1x_2 + F(x_1 + x_2) + G.$$

Magic change of variables

After long calculations and transformations Kowalevski gets

$$\begin{aligned} \frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} &= \frac{ds_1}{\sqrt{J(s_1)}} \\ -\frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} &= \frac{ds_2}{\sqrt{J(s_2)}} \end{aligned} \tag{4}$$

where

$$J(s) = 4s^3 + (c^2 - k^2 - 3l_1^2)s - l^2c^2 + l_1^3 - l_1k^2 + l_1c^2$$
$$R(x_i) = -x_i^4 + 6l_1x_i^2 + 4lcx_i + c^2 - k^2, \quad i = 1, 2$$

and s_1, s_2 are the roots of so called *Kowalevski's fundamental equation as a square equation in* s.

Kowalevski's fundamental equation

$$Q(s,x_1,x_2) := (x_1-x_2)^2(s-\frac{l_1}{2})^2 - R(x_1,x_2)(s-\frac{l_1}{2}) - \frac{1}{4}R_1(x_1,x_2) = 0 \tag{5}$$

satisfies discriminant separabilty condition

$$\mathcal{D}_s(Q)(x_1,x_2) = R(x_1)R(x_2)$$
$$\mathcal{D}_{x_1}(Q)(s,x_2) = J(s)R(x_2)$$
$$\mathcal{D}_{x_2}(Q)(s,x_1) = J(s)R(x_1)$$

with polynomials

$$J(s) = 4s^3 + (c^2-k^2-3l_1^2)s - l^2c^2 + l_1^3 - l_1k^2 + l_1c^2$$
$$R(x_i) = -x_i^4 + 6l_1x_i^2 + 4lcx_i + c^2 - k^2, \quad i = 1,2.$$

System of equations of the Kowalevski top may be rewritten as

$$\begin{aligned} 2\dot{x}_1 &= -if_1 \\ 2\dot{x}_2 &= if_2 \\ \dot{e}_1 &= -me_1 \\ \dot{e}_2 &= me_2 \\ 2\dot{r} &= i\left(e_2 - e_1 + x_1^2 - x_2^2\right) \\ 2c\dot{\gamma}_3 &= i\left(x_2e_1 - x_1e_2 + x_1x_2(x_2 - x_1)\right), \end{aligned} \tag{6}$$

where is

$$m = ir, \quad f_1 = rx_1 + c\gamma_3 \quad , f_2 = rx_2 + c\gamma_3,$$

and

$$f_i^2 = R(x_i) + e_i(x_1 - x_2)^2, \quad i = 1, 2.$$

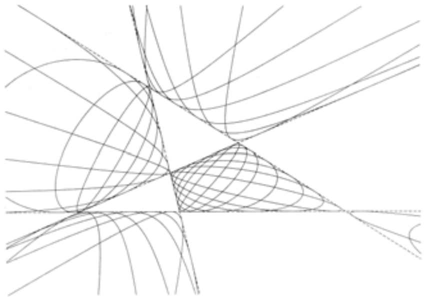
Two conics and tangential pencil

Starting with two conics C_1 and C_2 in general position, given by their tangential equations

$$C_1 \ : \ a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0$$

$$C_2 \ : \ w_2^2 - 4 w_1 w_3 = 0$$

Then, conics of the pencil $C(s) := C_1 + sC_2$ share four common tangents.



The coordinate equation of the conics of the pencil:

$$F(s,z_1,z_2,z_3) := \det M(s,z_1,z_2,z_3) = 0,$$

with matrix M:

$$M(s,z_1,z_2,z_3) = \begin{bmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & a_0 & a_1 & a_5-2s \\ z_2 & a_1 & a_2+s & a_3 \\ z_3 & a_5-2s & a_3 & a_4 \end{bmatrix}.$$

The point equation of the pencil $C(s)$ is then of the form of the quadratic polynomial in s

$$F := H + Ks + Ls^2 = 0$$

where H, K and L are quadratic expressions in z_1, z_2, z_3.

Equation of pencil $C_1 + sC_2$ in the Darboux coordinates

$$F(s, x_1, x_2) := L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2) = 0$$

Equation of pencil $C_1 + sC_2$ in the Darboux coordinates

$$F(s, x_1, x_2) := L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2) = 0$$

$$H(x_1, x_2) = (a_1^2 - a_0a_2)x_1^2x_2^2 + (a_0a_3 - a_5a_1)x_1x_2(x_1 + x_2)$$
$$+ (a_5^2 - a_0a_4)(x_1^2 + x_2^2) + (2(a_5a_2 - a_1a_3) + \frac{1}{2}(a_5^2 - a_0a_4)x_1x_2$$
$$+ (a_1a_4 - a_3a_5))(x_1 + x_2) + a_3^2 - a_2a_4$$
$$K(x_1, x_2) = -a_0x_1^2x_2^2 + 2a_1x_1x_2(x_1 + x_2) - a_5(x_1^2 + x_2^2)$$
$$- 4a_2x_1x_2 + 2a_3(x_1 + x_2) - a_4$$
$$L(x_1, x_2) = (x_1 - x_2)^2.$$

Theorem [V. Dragović, 2010]

- There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial F in s as a polynomial in x_1 and x_2 separates variables

$$\mathcal{D}_s(F)(x_1, x_2) = K^2 - 4LH = P(x_1)P(x_2).$$

- There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial F in x_2 as a polynomial in x_1 and s separates variables

$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s)P(x_1).$$

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$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s)P(x_1).$$

If all the zeros of the polynomial P are simple, then elliptic curves $\Gamma_1 : y^2 = P(x)$ and $\Gamma_2 : t^2 = J(s)$ are isomorphic and the later can be understood as a Jacobian of the former.

Discriminantly separable polynomials - definition

For a polynomial $\mathcal{F}(x_1, \dots, x_n)$ we say that it is discriminantly separable [V.Dragović CMP (2010)] if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i}\mathcal{F}(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is symmetrically discriminantly separable if

$$f_2 = f_3 = \cdots = f_n,$$

while it is strongly discriminantly separable if

$$f_1 = f_2 = f_3 = \cdots = f_n.$$

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Systems of Kowalevski type [V.D., K.K., RCD (2011)]

Given a discriminantly separable polynomial of the second degree in each of three variables

$$\mathcal{F}(x_1, x_2, s) := A(x_1, x_2)s^2 + 2B(x_1, x_2)s + C(x_1, x_2), \tag{7}$$

such that

$$\mathcal{D}_s(\mathcal{F})(x_1, x_2) = 4(B^2 - AC) = 4P(x_1)P(x_2),$$

and

$$\mathcal{D}_{x_1}(\mathcal{F})(s, x_2) = P(x_2)J(s)$$
$$\mathcal{D}_{x_2}(\mathcal{F})(s, x_1) = P(x_1)J(s).$$

Suppose, that a given system in variables x_1, x_2, e_1, e_2, r, γ_3, after some transformations reduces to

$$\dot{x}_1 = -if_1, \quad \dot{e}_1 = -me_1,$$
$$\dot{x}_2 = if_2, \quad \dot{e}_2 = me_2. \tag{8}$$

$$f_1^2 = P(x_1) + e_1 A(x_1, x_2), \quad f_2^2 = P(x_2) + e_2 A(x_1, x_2). \tag{9}$$

Suppose additionally, that the first integrals and invariant relations of the initial system reduce to a relation

$$P(x_2)e_1 + P(x_1)e_2 = C(x_1, x_2) - e_1 e_2 A(x_1, x_2). \tag{10}$$

Instead of (10) we can assume that

$$\dot{x}_1 \dot{x}_2 = -B(x_1, x_2) \tag{11}$$

where $B(x_1, x_2)$ is coefficient of polynomial (7).
If a system satisfies the above assumptions we will call it *a system of the Kowalevski type*.

Theorem V.D., K.K.

Given a system which reduces to (8, 9, 10). Then the system is linearized on the Jacobian of the curve

$$y^2 = J(z)(z-k)(z+k),$$

where J is a polynomial factor of the discriminant of $\mathcal{F}$ as a polynomial in x_1 and k is a constant such that

$$e_1 e_2 = k^2.$$

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Replacing the fundamental Kowalevski equation $Q(s, x_1, x_2) = 0$ by any discriminantly separable polynomial $F(x_1, x_2, s) = 0$ and with some additional assumption on the first integrals and invariant relations we obtained a new class of integrable systems - Kowalevski type systems.

The Sokolov system as a system of the Kowalevski type

Considered the Hamiltonian

$$\hat{H} = M_1^2 + M_2^2 + 2M_3^2 + 2c_1\gamma_1 + 2c_2(\gamma_2 M_3 - \gamma_3 M_2)$$

on $e(3)$ with the Lie-Poisson brackets

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \epsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0$$

Casimir functions: $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = a, \quad \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 = b.$
New variables:

$$z_1 = M_1 + iM_2, \quad z_2 = M_1 - iM_2,$$

$$e_1 = z_1^2 - 2c_1(\gamma_1 + i\gamma_2) - c_2(a + 2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_3 M_1 - \gamma_1 M_3)),$$
$$e_2 = z_2^2 - 2c_1(\gamma_1 - i\gamma_2) - c_2(a + 2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_1 M_3 - \gamma_3 M_1)).$$

The second integral of motion: $e_1 e_2 = k^2$.
Variables satisfy:

$$\dot{e}_1 = -4iM_3 e_1, \qquad \dot{e}_2 = 4iM_3 e_2$$
$$-\dot{z_1}^2 = P(z_1) + e_1(z_1 - z_2)^2,$$
$$-\dot{z_2}^2 = P(z_2) + e_2(z_1 - z_2)^2$$

where P is a polynomial of fourth degree given by

$$P(z) = -z^4 + 2Hz^2 - 8c_1 bz - k^2 + 4ac_1^2 - 2c_2^2(2b^2 - Ha) + c_2^4 a.$$

and

$$\dot{z_1} \cdot \dot{z_2} = -\left(F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2\right),$$
$$F(z_1, z_2) = -\frac{1}{2}\left(P(z_1) + P(z_2) + (z_1^2 - z_2^2)^2\right).$$

The Sokolov system is a system of the Kowalevski type. It can be explicitly integrated in the theta-functions of genus 2.

DSP for Sokolov case:

$$\tilde{F}(z_1,z_2,s)=(z_1-z_2)^2s^2+2\tilde{B}(z_1,z_2)s+\tilde{C}(z_1,z_2)$$

where

$$F^2(z_1,z_2)-P(z_1)P(z_2)=(z_1-z_2)^2C(z_1,z_2).$$

$$\tilde{C}(z_1,z_2)=C(z_1,z_2)+2F(z_1,z_2)(H+c_2^2a)+(H+c_2^2a)^2(z_1-z_2)^2$$

$$\tilde{B}(z_1,z_2)=F(z_1,z_2)+(H+c_2^2a)(z_1-z_2)^2.$$

Discriminants:

$$\mathcal{D}_s(\tilde{F})(z_1,z_2)=4P(z_1)P(z_2)$$

$$\mathcal{D}_{z_1}(\tilde{F})(s,z_2)=J(s)P(z_2),\ \mathcal{D}_{z_2}(Q)(s,z_1)=J(s)P(z_1).$$

Integration - generalized Kötter transformation:

For the polynomial $\tilde{F}(z_1, z_2, s)$ there exist polynomials $\alpha(z_1, z_2, s)$, $\beta(z_1, z_2, s)$, $f(s)$, $A_0(s)$ such that the following identity holds

$$\tilde{F}(z_1, z_2, s)A_0(s) = \alpha^2(z_1, z_2, s) + f(s)\beta(z_1, z_2, s).$$

The polynomials are defined by the formulae:

$$\begin{aligned}
A_0(u) &= 2s + 2H + 2c_2^2 a \\
B_0 &= -4c_1 b \\
f(s) &= 2s^3 + 2(H + 3c_2^2 a)s^2 + (-2k^2 + 8c_2^2 Ha + 8ac_1^2 - 8c_2^2 b^2 + 8c_2^4 a^2)s \\
&\quad + 4c_2^2 H^2 a - 2k^2 c_2^2 a + 8c_2^4 Ha^2 + 8ac_1^2 H - 2k^2 H - 8c_2^2 b^2 H + 4c_2^6 a^3 \\
&\quad + 8a^2 c_1^2 c_2^2 - 16c_1^2 b^2 - 8c_2^4 b^2 a \\
\alpha(z_1, z_2, s) &= A_0(s)(z_1 z_2 - s) + B_0(z_1 + z_2) + c_2^2 a A_0(s) \\
\beta(z_1, z_2, s) &= (z_1 + z_2)^2 - 2s - 2H - 2c_2^2 a.
\end{aligned}$$

Denote $\mathcal{F}(s) = \dfrac{\tilde{F}(z_1, z_2, s)}{(z_1 - z_2)^2}$ and consider the identity

$$\mathcal{F}(s) = \mathcal{F}(v) + (s - v)\mathcal{F}'(v) + (s - v)^2.$$

Then
$(s - v)^2(z_1 - z_2)^2 + (z_1 - z_2)^2(s - v)\mathcal{F}'(v) + \mathcal{F}(v)(z_1 - z_2)^2 = 0$
and from the last identities we get

$$(s - v)^2(z_1 - z_2)^2 + (s - v)\left(2v(z_1 - z_2)^2 + \tilde{B}(z_1, z_2)\right) + \alpha^2(z_1, z_2, v) + \beta(z_1, z_2, v) \cdot f(v) = 0.$$

The solutions s_1, s_2 of the last equation in s satisfy the following identity in v:

$$(s_1 - v)(s_2 - v) = \frac{\alpha^2(z_1, z_2, v)}{(z_1 - z_2)^2} + f(v)\frac{\beta(z_1, z_2, v)}{(z_1 - z_2)^2}.$$

Denote m_1, m_2, m_3 the zeros of the polynomial f, suppose they are real and $m_1 > m_2 > m_3$, and, following Kowalevski, introduce the functions

$$P_i = \sqrt{(s_1 - m_i)(s_2 - m_i)}.$$

The functions P_i satisfy

$$\begin{aligned} P_i &= \frac{\alpha(z_1, z_2, m_i)}{\sqrt{A_0(m_i)}(z_1 - z_2)} \\ &= \sqrt{A_0(m_i)}\frac{z_1 z_2 - m_i + c_2^2 a}{z_1 - z_2} + \frac{B_0(m_i)}{\sqrt{A_0(m_i)}}\frac{z_1 + z_2}{z_1 - z_2}. \end{aligned}$$

Introduce a more convenient notation:

$$X = \frac{z_1 z_2}{z_1 - z_2}, \qquad Y = \frac{1}{z_1 - z_2}, \qquad Z = \frac{z_1 + z_2}{z_1 - z_2}.$$

The quantities X,Y,Z satisfy the system of linear equations

$$X+\left(c_2^2a-m_1\right)Y+\frac{B_0}{A_0(m_1)}Z=\frac{P_1}{\sqrt{A_0(m_1)}}$$
$$X+\left(c_2^2a-m_2\right)Y+\frac{B_0}{A_0(m_2)}Z=\frac{P_2}{\sqrt{A_0(m_2)}}$$
$$X+\left(c_2^2a-m_3\right)Y+\frac{B_0}{A_0(m_3)}Z=\frac{P_3}{\sqrt{A_0(m_3)}}.$$

Then we get

$$M_2=\frac{1}{2iY}=\frac{i}{2}\frac{1}{\sum_{i=1}^3\frac{P_in_i}{f'(m_i)}},$$

$$M_1=\frac{Z}{2Y}=-\frac{n_1n_2n_3\sum_{i=1}^3\frac{P_in_jn_k}{f'(m_i)}}{4c_1B\sum_{i=1}^3\frac{P_in_i}{f'(m_i)}}.$$

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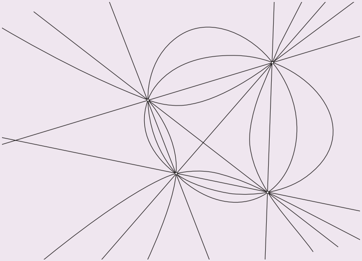
Classification of the strongly discriminantly separable polynomials type $\mathcal{P}_3^2$ up to gauge transformations [V.D.,K.K., (2011)]

Strongly discriminantly separable polynomials in three variables of degree two in each variable $\mathcal{F}(x_1, x_2, x_3)$ modulo gauge transformations $x_i \mapsto \frac{ax_i+b}{cx_i+d}$, $i = 1, 2, 3$ with corresponding pencils of conics are exhausted by the following list depending on distribution of roots of a non-zero polynomial $P(x)$:

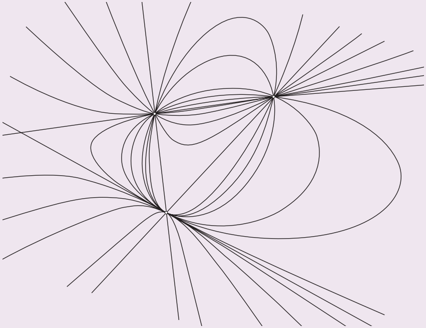
- four simple zeros $P(x) = (k^2x^2 - 1)(x^2 - 1)$,

$$\mathcal{F}_A = (-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2)\frac{x_3^2}{2} + (1-k^2)x_1x_2x_3 + \frac{1}{2}(x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1);$$



- one double and two simple zeros $P(x) = x^2 - e^2$, $e \neq 0$,

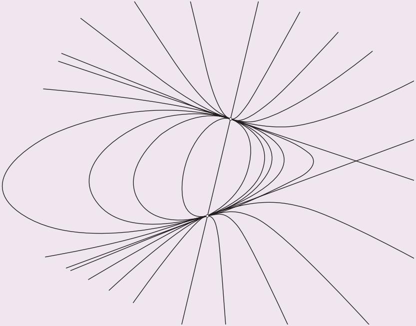
$$\mathcal{F}_B = x_1 x_2 x_3 + \frac{e}{2}(x_1^2 + x_2^2 + x_3^2 - e^2);$$



- two double zeros $P(x) = x^2$,

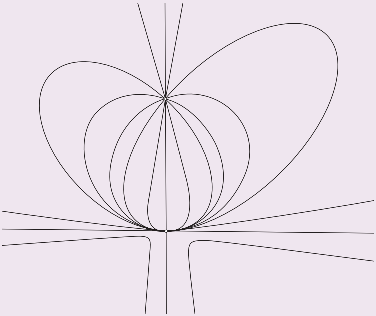
$$\mathcal{F}_{C1} = \lambda x_1^2 x_3^2 + \mu x_1 x_2 x_3 + \nu x_2^2, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{C2} = \lambda x_1^2 x_2^2 x_3^2 + \mu x_1 x_2 x_3 + \nu, \quad \mu^2 - 4\lambda\nu = 1;$$



- one simple and one triple zero $P(x) = x$,

$$\mathcal{F}_D = -\frac{1}{2}(x_1x_2 + x_2x_3 + x_1x_3) + \frac{1}{4}(x_1^2 + x_2^2 + x_3^2);$$



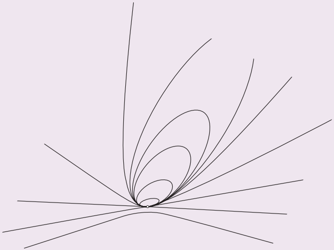
- one quadruple zero $P(x) = 1$,

$$\mathcal{F}_{E1} = \lambda(x_1+x_2+x_3)^2+\mu(x_1+x_2+x_3)+\nu, \quad \mu^2-4\lambda\nu = 1,$$

$$\mathcal{F}_{E2} = \lambda(x_2+x_3-x_1)^2+\mu(x_2+x_3-x_1)+\nu, \quad \mu^2-4\lambda\nu = 1,$$

$$\mathcal{F}_{E3} = \lambda(x_1+x_3-x_2)^2+\mu(x_1+x_3-x_2)+\nu, \quad \mu^2-4\lambda\nu = 1,$$

$$\mathcal{F}_{E4} = \lambda(x_1+x_2-x_3)^2+\mu(x_1+x_2-x_3)+\nu, \quad \mu^2-4\lambda\nu = 1.$$



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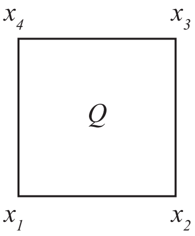
Adler-Bobenko-Suris (ABS) integrable quad-graphs

Consider two-dimensional lattice equations of the form $Q(x_1, x_2, x_3, x_4; \alpha, \beta) = 0$ where Q is linear in all four arguments.



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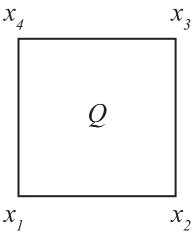

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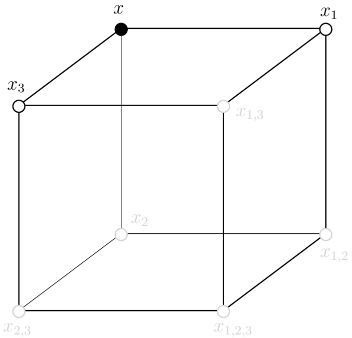

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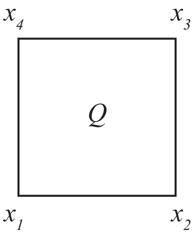

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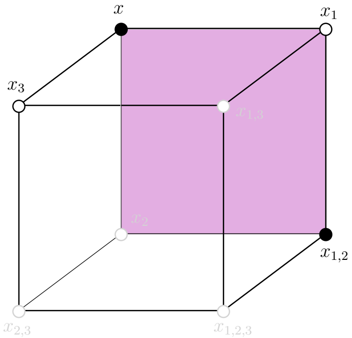

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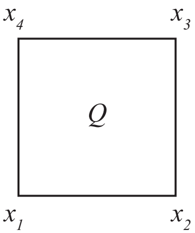

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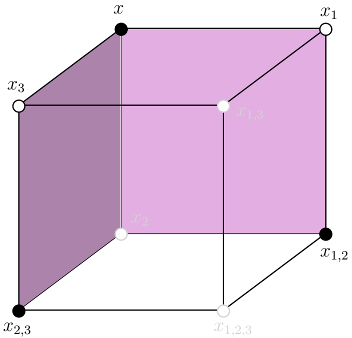

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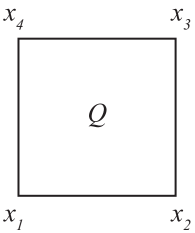

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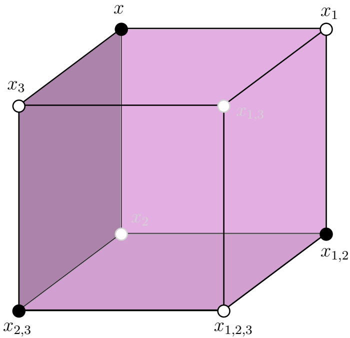

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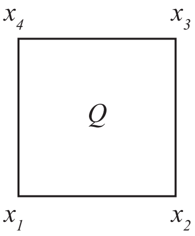

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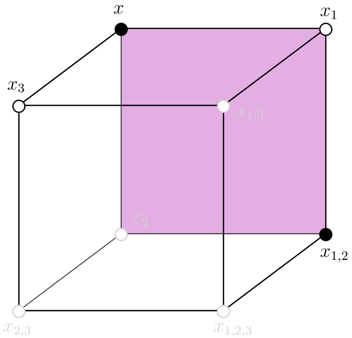

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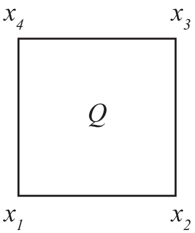

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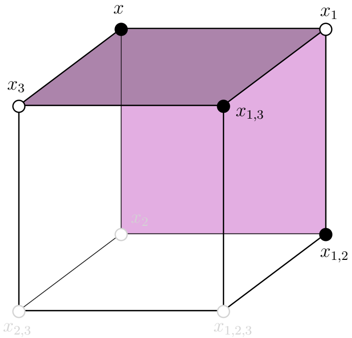

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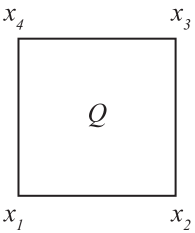

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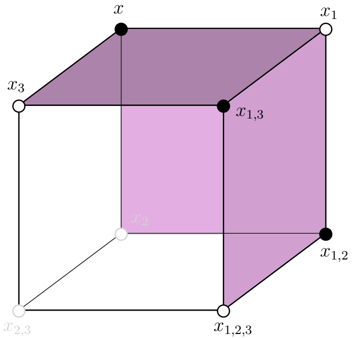

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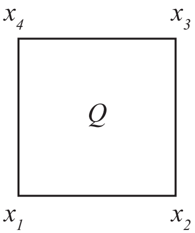

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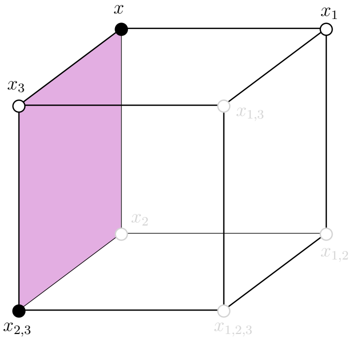

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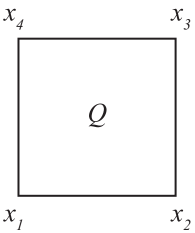

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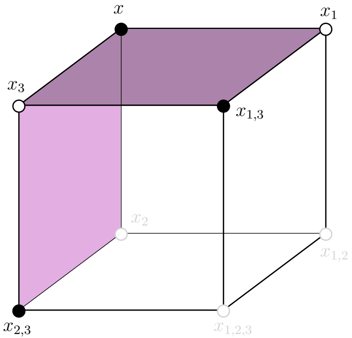

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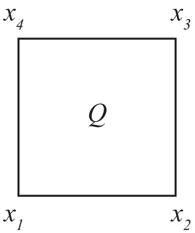

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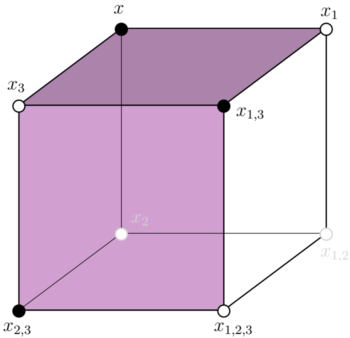

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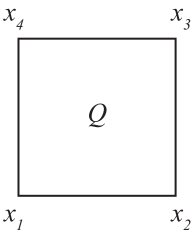

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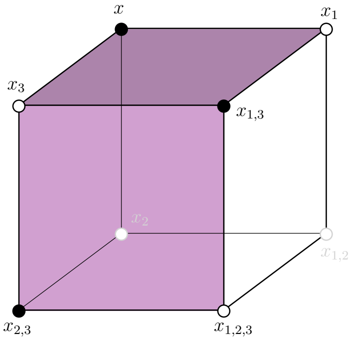

ABS 2009: "discriminant-like" operators

$$\mathcal{P}_4^1 \xrightarrow{\delta_{x_i,x_j}} \mathcal{P}_2^2 \xrightarrow{\delta_{x_k}} \mathcal{P}_1^4$$

$$h := \delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy},$$

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$$\hat{h}(x_1, x_2, \alpha) := \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}.$$

$$\frac{2Q_{x_1}}{Q} = \frac{h^{12}_{x_1} h^{34} - h^{14}_{x_1} h^{23} + h^{23} h^{34}_{x_3} - h^{23}_{x_3} h^{34}}{h^{12} h^{34} - h^{14} h^{23}}$$

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Thank you for your attention