Self-organised criticality

It's past and a recent field theory

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Outline

1. What is SOC?
   - SOC Models
2. The Manna Model
   - Manna Exponents in 1,2,3D
   - The Manna Model
3. Field theory
   - Simplifications
   - Diagrams
   - Tree level
   - The SOC mechanism

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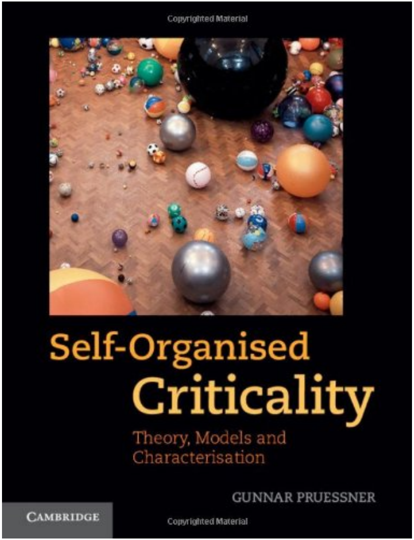
Preliminary Summary



- A brief reminder of **Self-Organised Criticality (SOC)**.
- An exact representation of the **Manna model** as a **field theory**.
- Results at **tree level**, *i.e.* the mean field theory of the Manna model (valid above the upper critical dimension)
- The field-theoretic **mechanism of SOC**.

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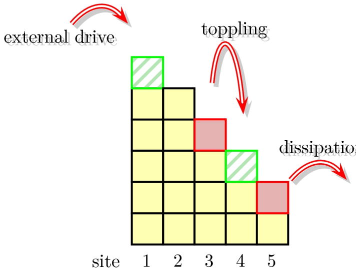
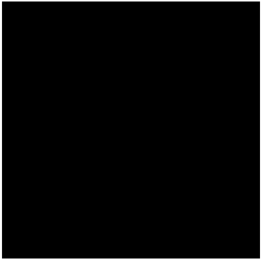
What is Self-Organised Criticality (SOC)?



The sandpile model:

- Bak, Tang and Wiesenfeld 1987.
- Simple (randomly driven) cellular automaton $\longrightarrow$ avalanches.
- Generates(?) scale invariant event statistics.
- **The physics of fractals.**

What is Self-Organised Criticality (SOC)?



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What is Self-Organised Criticality (SOC)?

SOC today: Non-trivial scale invariance (correlations!) in intermittent, interaction-dominated systems, like at a critical point, yet reached by self-tuning of a control parameter.

Key ingredients for SOC models:

- Separation of time scales (intermittency, avalanching).
- Non-linear interaction (thresholds).
- Observables: Correlation fcts., avalanche sizes and durations.
- Self-tuning to an ordinary critical point.
- **Scale invariance in space and time: Emergence! Universality!**

Universal (?) exponents τ and D

$$\mathcal{P}(s;L) = as^{-\tau}\mathcal{G}\left(\frac{s}{bL^{D}}\right)$$

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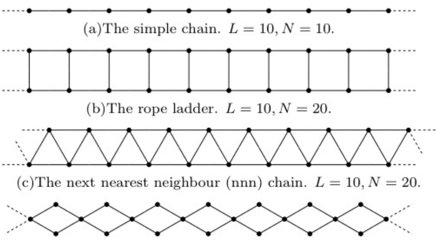
SOC Models

BUT: SOC Models notorious for **not** displaying systematic, robust, clean scaling behaviour. "Key ingredients" may not suffice.

Controversies: **Conservation**, **Stochasticity**, **Separation of time scales**, **Abelianness**.

Oslo Model and **Manna Model** both display **systematic**, **robust**, **clean** scaling behaviour:



(a)The simple chain. $L = 10, N = 10$.

(b)The rope ladder. $L = 10, N = 20$.

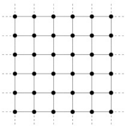
(c)The next nearest neighbour (nnn) chain. $L = 10, N = 20$.

(d)The Futatsubishi lattice. $L = 7, N = 22$.

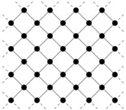
Same scaling exponents independent from lattice topology in $d = 1, 2, 3$ (From N Huynh, GP and Chew, 2011).

Manna on different lattices

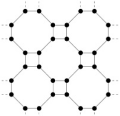
One and two dimensions



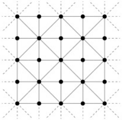
(a)The square lattice. $L_x = L_y = 6, N = 36$.



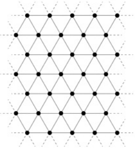
(b)The jagged lattice. $L_x = 4, L_y = 9, N = 36$.



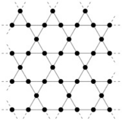
(c)The Archimedes lattice. $L_x = 8, L_y = 4, N = 32$.



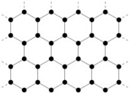
(d)The non-crossing (nc) diagonal square lattice. $L_x = L_y = 5, N = 25$.



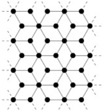
(a)The triangular lattice. $L_x = 5, L_y = 7, N = 35$.



(b)The Kagomé lattice. $L_x = 10, L_y = 4, N = 40$.



(c)The honeycomb lattice. $L_x = 9, L_y = 4, N = 36$.



(d)The Mitsubishi lattice. $L_x = 5, L_y = 7, N = 35$.

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

Manna on different lattices

One and two dimensions

lattice	d	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
simple chain	1	2.27(2)	1.117(8)	1.450(12)	1.19(2)	0.998(4)	1.260(13)	2.000(4)	0.27(2)	0.27(3)	0.259(14)
rope ladder	1	2.24(2)	1.108(9)	1.44(2)	1.18(3)	0.998(7)	1.26(2)	1.989(5)	0.24(2)	0.26(5)	0.26(2)
nnn chain	1	2.33(11)	1.14(4)	1.48(11)	1.22(14)	0.997(15)	1.27(5)	1.991(11)	0.33(11)	0.3(2)	0.27(5)
Futatsubishi	1	2.24(3)	1.105(14)	1.43(3)	1.16(6)	0.999(15)	1.24(5)	2.008(11)	0.24(3)	0.23(9)	0.24(5)
square	2	2.748(13)	1.272(3)	1.52(2)	1.48(2)	1.992(8)	1.380(8)	1.9975(11)	0.748(13)	0.73(4)	0.76(2)
jagged	2	2.764(15)	1.276(4)	1.54(2)	1.49(3)	1.995(7)	1.384(8)	2.0007(12)	0.764(15)	0.76(5)	0.77(2)
Archimedes	2	2.76(2)	1.275(6)	1.54(3)	1.50(3)	1.997(10)	1.382(11)	2.001(2)	0.76(2)	0.78(6)	0.76(3)
nc diagonal square	2	2.750(14)	1.273(4)	1.53(2)	1.49(2)	1.992(7)	1.381(8)	2.0005(12)	0.750(14)	0.75(4)	0.76(2)
triangular	2	2.76(2)	1.275(5)	1.51(2)	1.47(3)	2.003(11)	1.388(12)	1.997(2)	0.76(2)	0.71(6)	0.78(3)
Kagomé	2	2.741(13)	1.270(4)	1.53(2)	1.49(2)	1.993(8)	1.381(9)	1.9994(12)	0.741(13)	0.75(5)	0.76(2)
honeycomb	2	2.73(2)	1.268(6)	1.55(4)	1.51(4)	1.990(13)	1.376(14)	2.000(2)	0.73(2)	0.79(8)	0.75(3)
Mitsubishi	2	2.75(2)	1.273(6)	1.54(3)	1.50(4)	1.999(12)	1.387(12)	1.998(2)	0.75(2)	0.77(7)	0.77(3)

From: Huynh, G P, Chew, 2011

The Manna Model has been investigated numerically in great detail.

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Manna on different lattices

Three dimensions

Lattice	$\overline{q}$	$\overline{q^{(v)}}$	$\langle z \rangle$	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SC	6	1	[0.622325(1)]	3.38(2)	1.408(3)	1.779(7)	1.784(9)	3.04(5)	1.45(4)	2.0057(5)	1.38(2)	1.395(16)	1.36(13)
BCC	8	4	[0.600620(2)]	3.36(2)	1.404(4)	1.777(8)	1.78(1)	2.99(2)	1.444(18)	2.0030(5)	1.36(2)	1.390(19)	1.33(6)
BCCN	14	5	[0.581502(1)]	3.38(3)	1.408(4)	1.776(9)	1.783(11)	3.01(3)	1.44(3)	2.0041(6)	1.38(3)	1.39(2)	1.32(7)
FCC	12	4	[0.589187(3)]	3.35(4)	1.402(8)	1.765(16)	1.78(2)	3.1(2)	1.48(14)	2.0035(11)	1.35(4)	1.37(4)	1.5(5)
FCCN	18	5	[0.566307(3)]	3.38(4)	1.408(7)	1.781(14)	1.787(18)	3.00(4)	1.44(3)	2.0051(8)	1.38(4)	1.40(3)	1.32(9)
Overall				3.370(11)	1.407(2)	1.777(4)	1.783(5)	3.003(14)	1.442(12)	2.0042(3)		1.380(13)	

Fractals

Lattice	d	d_w	D	τ	z	α	D_a	τ_a	$\mu_1^{(s)}$	$-\Sigma_s$	$-\Sigma_t$	$-\Sigma_a$
SSTK	1.464..	2.552..	2.94(3)	1.13(2)	1.817(17)	1.21(2)	1.466(5)	1.273(11)	2.551(6)	0.37(6)	0.38(4)	0.399(17)
ARRO	1.584..	2.322..	2.793(2)	1.173(2)	1.673(1)	1.280(2)	1.5847(3)	1.2985(6)	2.3103(4)	0.484(5)	0.468(3)	0.473(1)
CRAB	1.584..	2.578..	3.020(5)	1.151(4)	1.837(3)	1.237(4)	1.5847(8)	1.279(2)	2.5655(12)	0.456(11)	0.435(7)	0.443(3)
SITE	2	2.584..	3.232(6)	1.211(4)	1.870(4)	1.357(4)	1.9975(9)	1.339(2)	2.5533(6)	0.682(14)	0.667(8)	0.677(3)
EXGA	2.584..	2.321..	3.352(4)	1.312(3)	1.835(3)	1.581(3)	2.5895(6)	1.3915(8)	2.3000(2)	1.046(10)	1.066(6)	1.014(2)

From: Huynh, G P, 2012

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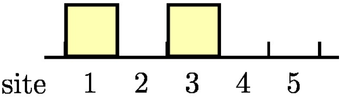
The Manna Model

Manna 1991, Dhar 1999



Manna Model (1991)

- Critical height model.
- Stochastic.
- Bulk drive.
- Robust, solid, universal, reproducible.
- Defines a universality class.

The Manna Model

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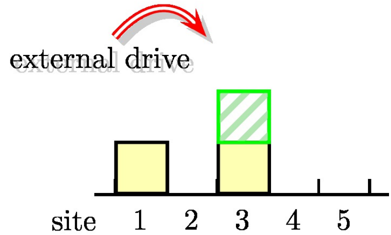

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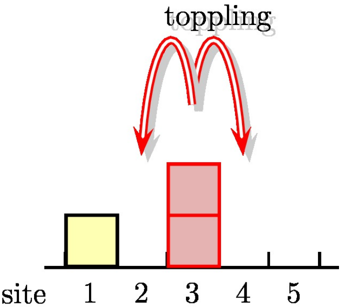

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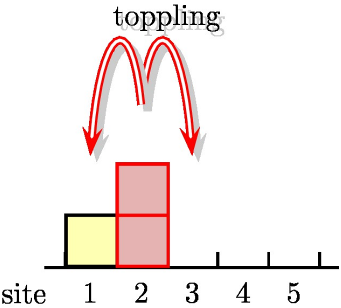

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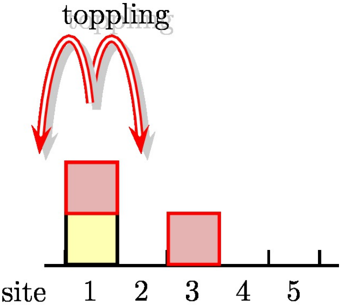

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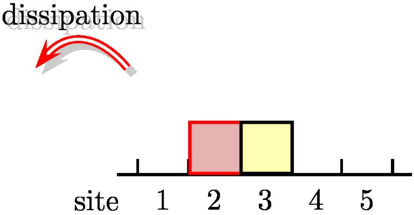

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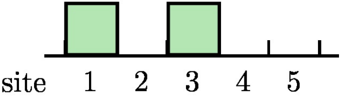
The Manna Model

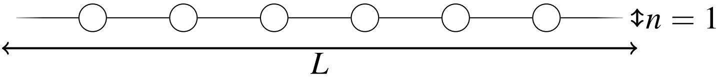
Revised version



Problem: Manna Model appears to be **excluded volume ("fermionic")** — don't smooth out!
At most one particle per site.
Solution: Introduce **carrying capacity n** and make toppling probabilistic (occupation over **n**).

The Manna Model

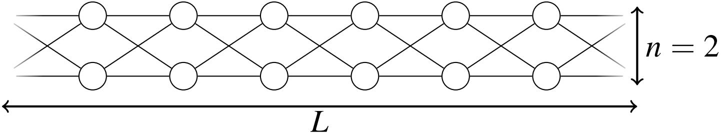
Revised and generalised version



- One dimensional lattice, length L, carrying capacity n.
- Sites within each column equivalent (**particles per column**).
- At relaxation, probability to hit a neighbouring, occupied site is occupation over carrying capacity n.
- Field-theory now easy (Manna's fermionicity is "spurious").
- Manna Model with carrying capacity = Manna Model on $L \times n$ lattice.

The Manna Model

Revised and generalised version



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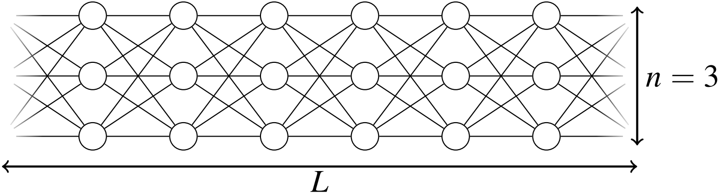

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The Manna Model

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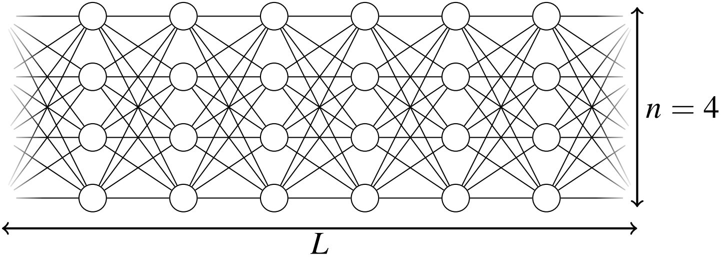

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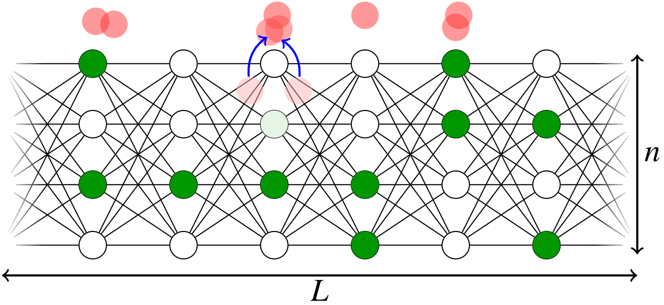
The Manna Model

Revised and generalised version (EXACT!)



- Substrate particles
- Active particles
- Poissonian diffusion
- ...deposition or
- ...activation.

- Operators: $\sigma^\dagger(\mathbf{x})$, $\sigma(\mathbf{x})$
- Operators: $a^\dagger(\mathbf{x})$, $a(\mathbf{x})$
- $(a^\dagger(\mathbf{y}) - a^\dagger(\mathbf{x}))a(\mathbf{x})$
- $\sigma^\dagger(\mathbf{y})(1 - \frac{1}{n}\sigma^\dagger(\mathbf{y})\sigma(\mathbf{y}))a(\mathbf{x})$
- $\frac{1}{n}\left(a^\dagger(\mathbf{y})\right)^2\sigma(\mathbf{y})a(\mathbf{x})$

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3 Field theory
- Simplifications
- Diagrams
- Tree level
- The SOC mechanism

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Effective Field Theory

Simplifications:

- Take continuum limit and apply cylindrical boundary conditions.
- Apply Doi shift and remove irrelevant terms.
- Substrate density shifted to be excess above $n/2$.

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Effective Field Theory

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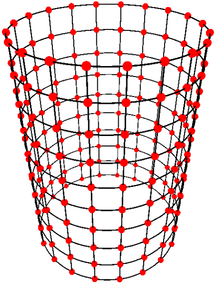

Open boundaries needed for stationarity. Cylindrical boundary conditions simplify bare propagator:

$$\frac{1}{-\imath\omega + D(\mathbf{k}^2 + q_n{}^2)}$$

where $q_n = \frac{\pi}{L} n$ with $n = 1, 2, \ldots$
Lowest mode, $q_1 = \pi/L$, controls phase transition.

Average avalanche size in d dimension: $\langle s \rangle_d = d \langle s \rangle_1$.

- Apply Doi shift and remove irrelevant terms.
- Substrate density shifted to be excess above $n/2$.

Effective Field Theory

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Effective Field Theory

Simplifications:

- Take continuum limit and apply cylindrical boundary conditions.
- Apply Doi shift and remove irrelevant terms.
- Substrate density shifted to be excess above $n/2$.
  Branching ratio 1 everywhere if interaction with substrate can be ignored.

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Effective Field Theory

Liouvillian

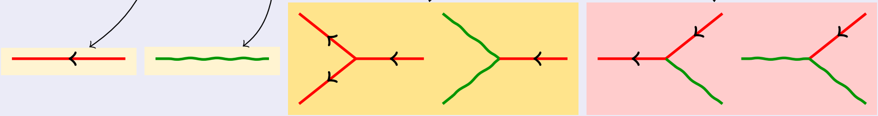
Path integral

$$\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}\psi\mathcal{D}\psi^* e^{-\int \mathrm{d}^d x \mathrm{d}t\, \mathcal{L}}$$

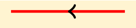
whith Liouvillian

$$\mathcal{L} = \phi^*(\partial_t - D\nabla^2)\phi + \psi^*\partial_t\psi + D\left(\phi^{*2} - \psi^{*2}\right)\phi + D\lambda\left(\phi^* - \psi^*\right)\psi\phi$$

Diagrams:



Diagrammatic ingredients: Bare propagators

 $$= \frac{\delta_{nm}\bar{\delta}(\omega' - \omega)\bar{\delta}(\mathbf{k}' - \mathbf{k})}{-\imath\omega + D(\mathbf{k}^2 + q_n{}^2)}$$

Activity propagator

- Due to dissipation at boundaries, eigensystem $\sqrt{2/L}\sin(q_n z)$.
- **Lowest mode, $q_1 = \pi/L$, controls phase transition.**
- Lack of orthogonality (as in critical Casimir systems): $\int dz\, \sin(q_n z)\sin(q_m z)\sin(q_l z) \neq \delta_{n+m+l,0}$
- Thus $\sum_{nml}$, no momentum conservation at vertices.

 $$= \frac{\delta_{nm}\bar{\delta}(\omega' - \omega)\bar{\delta}(\mathbf{k}' - \mathbf{k})}{-\imath\omega + \epsilon}$$

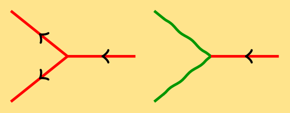
Substrate deposition

- Deposition, no diffusion.
- Causality restored by $1 \gg \epsilon \neq 0$.

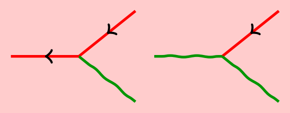
Diagrammatic ingredients: Vertices

The (effective) interaction vertices are

- Spontaneous **branching** and substrate deposition:



- **Annihilation**: Substrate interaction resulting in attenuation or deposition:

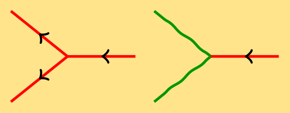


All relevant for $d \leqslant d_c = 4$. Loops occur.

Diagrammatic ingredients: Vertices

The (effective) interaction vertices are

- Spontaneous **branching** and substrate deposition:



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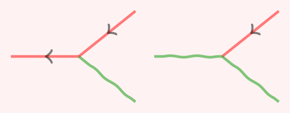

Only the former are relevant for $d > d_c = 4$; as in ϕ^4 the latter enter only for the lowest mode. No loops.

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Tree level — applies above $d_c = 4$

Average avalanche size

Tree level becomes exact above $d_c = 4$.
Vertices do not contribute to $\left\langle a(\mathbf{x}, z, t) a^\dagger(\mathbf{x}_0, z_0, 0) \right\rangle = \longleftarrow$.
Avalanche is (half) time and space integrated activity ($\mathbf{k}_0 = \mathbf{k} = 0$ in periodic subspace):

$$2\langle s \rangle = \underbrace{\frac{1}{L}\int \mathrm{d}z_0 \int \mathrm{d}^{d-1}x_0}_{\text{uniform drive}} \int_0^\infty \mathrm{d}t \, \left\langle a(\mathbf{x}, z, t) a^\dagger(\mathbf{x}_0, z_0, 0) \right\rangle$$

$$= \frac{1}{L} \sum_{n \text{odd}} \frac{2}{q_n} \frac{2}{q_n} \frac{2}{L} (D q_n^2)^{-1} = \frac{L^2}{12D}$$

Same as average avalanche size from random walkers: First hint of non-renormalisation of propagator (at $\omega = 0$).

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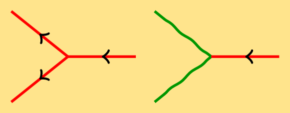
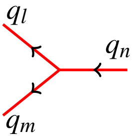
Tree level — applies above $d_c = 4$

Tree level becomes exact above $d_c = 4$. Two vertices are present:



Higher orders:

$$4\left\langle s^2\right\rangle = \overbrace{2}^{\text{symmetry factor}} \frac{1}{L}\left(\frac{2}{L}\right)^3 \sum_{\substack{n,m,l\\ \text{odd}}} \frac{4}{q_l q_m}$$



$$\frac{2}{q_n} = \frac{L^6}{560D^2}$$

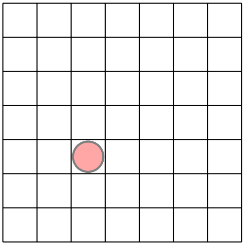
Similarly for higher order moments...

Tree level — applies above $d_c = 4$

Underlying process

Physics of the tree level diagrams (Manna Model above $d_c = 4$):

The **mean field theory of the Manna Model** is a **fair branching random walk on a lattice with open boundaries**.



In contrast to the usual *effective* mean-field theory of, the above identifies precisely which correlations and fluctuations are to be ignored.

Tree level — applies above $d_c = 4$

Underlying process

Physics of the tree level diagrams (Manna Model above $d_c = 4$):

Mean field theory of the Manna Model is a
fair branching random walk on a lattice with open boundaries.

Avalanche moments can be calculated exactly.[1] Compare universal moment ratios to numerics at $d = 5$ (GP and Nguyen Huynh):

Observable	analytical	numerical (leading order)
$\langle s \rangle$	$(d/6)L^2 = 0.833\ldots L^2$	$0.83334(6)L^2$
$\langle s^3 \rangle \langle s \rangle / \langle s^2 \rangle^2$	$3.08754\ldots$	$3.061(5)$
$\langle s^4 \rangle \langle s^2 \rangle / \langle s^3 \rangle^2$	$1.6693\ldots$	$1.65(2)$
$\langle s^5 \rangle \langle s^3 \rangle / \langle s^4 \rangle^2$	$1.4005\ldots$	$1.38(3)$

[1] Tedious! Use Mathematica!

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The SOC mechanism

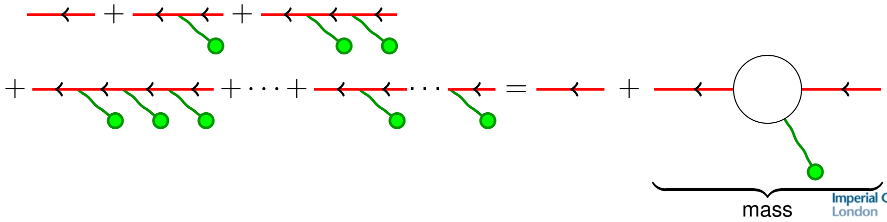
How does SOC work?

At criticality the renormalised mass vanishes:

$$= \frac{\delta_{nm}\bar{\delta}(\omega' - \omega)\bar{\delta}(\mathbf{k}' - \mathbf{k})}{-\imath\omega + D(\mathbf{k}^2 + q_n^2) + r_0}$$

$\longrightarrow$ Why are the propagators massless?

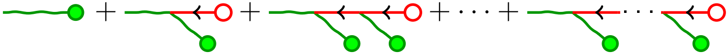
Mass is attenuation (loss of activity). At tree level:



The SOC mechanism

How does SOC work?

Attenuation leads to deposition by the external drive — diagrams have that symmetry.
Density of particles in the substrate:



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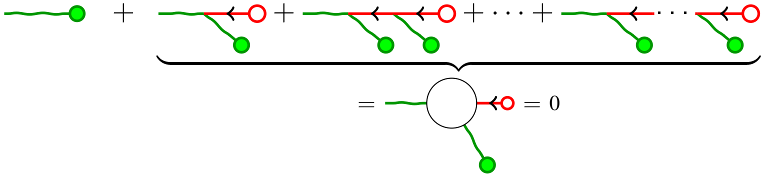

Additional deposition by external drive vanishes at stationarity.

The SOC mechanism

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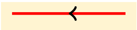
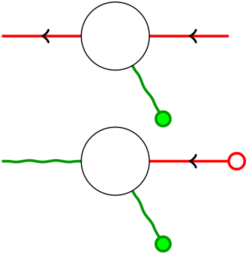


$$= \frac{\delta_{nm}\bar{\delta}(\omega' - \omega)\bar{\delta}(\mathbf{k}' - \mathbf{k})}{-\imath\omega + D(\mathbf{k}^2 + q_n^2) + r_0}$$

Propagator renormalisation including mass:

Additional deposition:

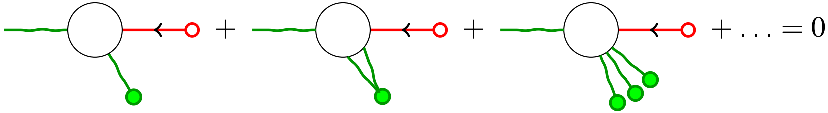


$= \quad 0$

Only difference between the two diagrams: Left most vertex (coupling identical at renormalised and bare level).

The SOC mechanism
Beyond tree level

Argument extends beyond tree level and beyond one-point correlators of the substrate:



Propagator does not renormalise at any order.

This is why the bare propagator gives the exact average avalanche size as derived via random walker approach.

Correlation in inactive particles are weak, activity is where SOC shows.

The SOC mechanism

So how does it work then?

Symmetry of vertices and stationarity.

- Mass is attenuation of activity.
- Conservation links attenuation to (additional) substrate deposition...
- or equivalently, symmetry of vertices equates mass terms of activity and substrate deposition terms.
- Additional substrate deposition vanishes *as we choose to consider stationarity.*
- **Substrate organises to the unique critical (massless, stationary) state** (independent of mode of driving).
- The activity propagator is not renormalised at any order.

What are the key findings?

- **Field theory for the Manna Model derived from microscopic rules**.
- Now we know **why** and **how** the propagator is massless.
- **Symmetry of vertices**, reflecting conservation (**conservation not necessary!**),
- ... ensures that the renormalisation of the propagator vanishes at **stationarity**.
- Criticality (masslessness) **regardless of mode of driving**.
- Correlations during an avalanche are non-trivial and shift the **local branching ratio**.
- Correlations in the substrate are weak (possibly irrelevant).
- Other mechanisms challenged: Absorbing states, sweeping across the critical point, Goldstone bosons, no criticality ...

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Interesting technical questions

There are a number of interesting technical features in this field theory:

- Renormalisation for Doi-Pelitti field theories.
- Excluded volume ("fermionicity").
- Surfaces, *i.e.* finite lattice (lack of conservation in vertices).

Exactly solvable, accessible by the same techniques, great fun:

The Wiener Sausage Problem (with branching)

Thank you!