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Riemann-Hilbert Problems, families of commuting operators and soliton equations

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PLAN

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Based on:

- V. S. Gerdjikov, D. J. Kaup. *Reductions of* 3×3 *polynomial bundles and new types of integrable* 3–*wave interactions.* In Nonlinear evolution equations: integrability and spectral methods, Ed. A. P. Fordy, A. Degasperis, M. Lakshmanan, Manchester University Press, (1981), p. 373–380

- V. S. Gerdjikov. On new types of integrable 4-wave interactions. AIP Conf. proc. **1487** pp. 272-279; (2012).

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- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).

- V. S. Gerdjikov, A. B. Yanovski *On soliton equations with* $\mathbb{Z}_h$

and $\mathbb{D}_h$ *reductions: conservation laws and generating operators.* J. Geom. Symmetry Phys. **31**, 57–92 (2013).

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RHP with canonical normalization

$$\xi^+(\vec{x},t,\lambda) = \xi^-(\vec{x},t,\lambda)G(\vec{x},t,\lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^+(\vec{x},t,\lambda) = \mathbb{1},$$

$\xi^{\pm}(\vec{x},t,\lambda) \in \mathfrak{G}$

Consider particular type of dependence $G(\vec{x},t,\lambda)$:

$$i\frac{\partial G}{\partial x_s} - \lambda^k [J_s, G(\vec{x},t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k [K, G(\vec{x},t,\lambda)] = 0.$$

where $J_s \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP:

$$\xi^{\pm}(\vec{x},t,\lambda) = \exp Q(\vec{x},t,\lambda), \qquad Q(\vec{x},t,\lambda) = \sum_{k=1}^{\infty} Q_k(\vec{x},t)\lambda^{-k}.$$

where all $Q_k(\vec{x},t) \in \mathfrak{g}$. However,

$$\mathcal{J}_s(\vec{x},t,\lambda) = \xi^{\pm}(\vec{x},t,\lambda) J_s \hat{\xi}^{\pm}(\vec{x},t,\lambda), \qquad \mathcal{K}(\vec{x},t,\lambda) = \xi^{\pm}(\vec{x},t,\lambda) K \hat{\xi}^{\pm}(\vec{x},t,\lambda),$$

belong to the algebra $\mathfrak{g}$ for any J and K from $\mathfrak{g}$. If in addition K also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}_s(\vec{x},t,\lambda),\mathcal{K}(\vec{x},t,\lambda)]=0.$$

Zakharov-Shabat theorem

Theorem 1. *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables $\vec{x}$ and t as above. Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$L_s\xi^{\pm}\equiv i\frac{\partial\xi^{\pm}}{\partial x_s}+U_s(\vec{x},t,\lambda)\xi^{\pm}(\vec{x},t,\lambda)-\lambda^k[J_s,\xi^{\pm}(\vec{x},t,\lambda)]=0,$$
$$M\xi^{\pm}\equiv i\frac{\partial\xi^{\pm}}{\partial t}+V(\vec{x},t,\lambda)\xi^{\pm}(\vec{x},t,\lambda)-\lambda^k[K,\xi^{\pm}(\vec{x},t,\lambda)]=0.$$

Proof. Introduce the functions:

$$g_s^{\pm}(\vec{x},t,\lambda)=i\frac{\partial\xi^{\pm}}{\partial x_s}\hat{\xi}^{\pm}(\vec{x},t,\lambda)+\lambda^k\xi^{\pm}(\vec{x},t,\lambda)J_s\hat{\xi}^{\pm}(\vec{x},t,\lambda),$$
$$g^{\pm}(\vec{x},t,\lambda)=i\frac{\partial\xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(\vec{x},t,\lambda)+\lambda^k\xi^{\pm}(\vec{x},t,\lambda)K\hat{\xi}^{\pm}(\vec{x},t,\lambda),$$

and using

$$i\frac{\partial G}{\partial x_s} - \lambda^k [J_s, G(\vec{x}, t, \lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k [K, G(\vec{x}, t, \lambda)] = 0.$$

prove that

$$g_s^+(\vec{x}, t, \lambda) = g_s^-(\vec{x}, t, \lambda), \qquad g^+(\vec{x}, t, \lambda) = g^-(\vec{x}, t, \lambda),$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g_s^+(\vec{x}, t, \lambda) = \lambda^k J_s, \qquad \lim_{\lambda\to\infty} g^+(\vec{x}, t, \lambda) = \lambda^k K.$$

and make use of Liouville theorem to get

$$g_s^+(\vec{x}, t, \lambda) = g_s^-(\vec{x}, t, \lambda) = \lambda^k J_s - \sum_{l=1}^{k} U_{s;l}(\vec{x}, t)\lambda^{k-l},$$

$$g^+(\vec{x}, t, \lambda) = g^-(\vec{x}, t, \lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(\vec{x}, t)\lambda^{k-l}.$$

We shall see below that the coefficients $U_{s;l}(\vec{x},t)$ and $V_l(\vec{x},t)$ can be expressed in terms of the asymptotic coefficients Q_s of $\xi^{\pm}(\vec{x},t,\lambda)$.

Now remember the definition of $g_s^{+}(\vec{x},t,\lambda)$

$$\begin{aligned} g_s^{\pm}(\vec{x},t,\lambda) &= i\frac{\partial\xi^{\pm}}{\partial x_s}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda^k\xi^{\pm}(\vec{x},t,\lambda)J_s\hat{\xi}^{\pm}(\vec{x},t,\lambda) \\ &= \lambda^k J_s - \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}, \end{aligned}$$

Multiply both sides by $\xi^{\pm}(\vec{x},t,\lambda)$ and move all the terms to the left:

$$i\frac{\partial\xi^{\pm}}{\partial x_s} + \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}\xi^{\pm}(\vec{x},t,\lambda) - \lambda^k[J_s,\xi^{\pm}(\vec{x},t,\lambda)] = 0,$$

i.e. $L_s\xi^{\pm}(\vec{x},t,\lambda) = 0$. □

Lemma 1. *The set of operators L_s and M commute*

$$[L_s, L_j] = 0, \qquad [L_s, M] = 0,$$

i.e. the following set of equations hold:

$$i\frac{\partial U_s}{\partial x_j} - i\frac{\partial U_j}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, U_j(\vec{x},t,\lambda) - \lambda^k J_j] = 0,$$
$$i\frac{\partial U_s}{\partial t} - i\frac{\partial V}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, V(\vec{x},t,\lambda) - \lambda^k K] = 0.$$

where

$$U_s(\vec{x},t,\lambda) = \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}, \qquad V(\vec{x},t,\lambda) = \sum_{l=0}^{k} V_l(\vec{x},t)\lambda^{k-l}.$$

Jets of order k

How to parametrize $U_s(\vec{x},t,\lambda)$ and $V(\vec{x},t,\lambda)$?
Use:

$$\xi^{\pm}(\vec{x},t,\lambda) = \exp Q(\vec{x},t,\lambda), \qquad Q(\vec{x},t,\lambda) = \sum_{k=1}^{\infty} Q_k(\vec{x},t)\lambda^{-k}.$$

and consider the jets of order k of $\mathcal{J}(x,\lambda)$ and $\mathcal{K}(x,\lambda)$:

$$\mathcal{J}_s(\vec{x},t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(\vec{x},t,\lambda) J_l \hat{\xi}^{\pm}(\vec{x},t,\lambda)\right)_+ = \lambda^k J_s - U_s(\vec{x},t,\lambda),$$
$$\mathcal{K}(\vec{x},t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(\vec{x},t,\lambda) K \hat{\xi}^{\pm}(\vec{x},t,\lambda)\right)_+ = \lambda^k K - V(\vec{x},t,\lambda).$$

Express $U_s(x) \in \mathfrak{g}$ in terms of $Q_s(x)$:

$$\mathcal{J}_s(\vec{x},t,\lambda) = J_s + \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{ad}_Q^k J_s, \qquad \mathcal{K}(\vec{x},t,\lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{ad}_Q^k K,$$
$$\operatorname{ad}_Q Z = [Q, Z], \qquad \operatorname{ad}_Q^2 Z = [Q,[Q,Z]], \quad \ldots$$

and therefore for $U_{s;l}$ we get:

$$U_{s;1}(\vec{x},t) = -\mathrm{ad}\,_{Q_1} J_s, \qquad U_{s;2}(\vec{x},t) = -\mathrm{ad}\,_{Q_2} J_s - \frac{1}{2}\mathrm{ad}\,^2_{Q_1} J_s$$

$$U_{s;3}(\vec{x},t) = -\mathrm{ad}\,_{Q_3} J_s - \frac{1}{2}\left(\mathrm{ad}\,_{Q_2}\mathrm{ad}\,_{Q_1} + \mathrm{ad}\,_{Q_1}\mathrm{ad}\,_{Q_2}\right) J_s - \frac{1}{6}\mathrm{ad}\,^3_{Q_1} J_s$$

$$\vdots$$

$$U_{s;k}(\vec{x},t) = -\mathrm{ad}\,_{Q_k} J_s - \frac{1}{2}\sum_{s+p=k} \mathrm{ad}\,_{Q_s}\mathrm{ad}\,_{Q_p} J_s - \frac{1}{6}\sum_{s+p+r=k} \mathrm{ad}\,_{Q_s}\mathrm{ad}\,_{Q_p}\mathrm{ad}\,_{Q_r} J_s - \cdots - \frac{1}{k!}\mathrm{ad}\,^k_{Q_1} J_s,$$

and similar expressions for $V_l(\vec{x},t)$ with J_s replaced by K.

Reductions of polynomial bundles

$$\begin{aligned}
&\text{a)} \qquad A\xi^{+,\dagger}(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^{-}(x,t,\lambda), \qquad AQ^{\dagger}(x,t,\epsilon\lambda^*)\hat{A} = -Q(x,t,\lambda),\\
&\text{b)} \qquad B\xi^{+,*}(x,t,\epsilon\lambda^*)\hat{B} = \xi^{-}(x,t,\lambda), \qquad BQ^{*}(x,t,\epsilon\lambda^*)\hat{B} = Q(x,t,\lambda),\\
&\text{c)} \qquad C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^{-}(x,t,\lambda), \qquad CQ^{\dagger}(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda),
\end{aligned}$$

where $\epsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the $\mathbb{Z}_N$-reductions we may have:

$$D\xi^{\pm}(x,t,\omega\lambda)\hat{D} = \xi^{\pm}(x,t,\lambda), \qquad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda),$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

On N-wave equations – $k = 1$

Lax representation involves two Lax operators linear in λ:

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(\vec{x}, t, \lambda) - \lambda[J, \xi^{\pm}(\vec{x}, t, \lambda)] = 0,$$

$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q(x,t)]\xi^{\pm}(\vec{x}, t, \lambda) - \lambda[K, \xi^{\pm}(\vec{x}, t, \lambda)] = 0.$$

The corresponding equations take the form:

$$i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[K, \frac{\partial Q}{\partial x}\right] - [[J, Q], [K, Q(x,t)]] = 0$$

$$Q(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad \begin{array}{r} J = \mathrm{diag}\,(a_1, a_2, a_3), \\ K = \mathrm{diag}\,(b_1, b_2, b_3), \end{array}$$

Then the 3-wave equations take the form:

$$\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0,$$
$$\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0,$$
$$\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0,$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

For 3-dimensional space-time we consider Q of the form (), but now let u_j and v_j be functions of $x_1 = x$, $x_2 = y$ and t. Let also $J_1 = J$ and $J_2 = I = \operatorname{diag}(c_1, c_2.c_3)$. Now the corresponding solution of the RHP $\xi^{\pm}(x, y, t, \lambda)$ will be FAS not only of L and M above, but also of

$$P\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial y} + [I, Q(x,t)]\xi^{+}(\vec{x}, t, \lambda) - \lambda[I, \xi^{+}(\vec{x}, t, \lambda)] = 0,$$

and all these three operators will mutually commute, i.e. along with $[L, M] = 0$ we will have also $[L, P] = 0$ and $[P, M] = 0$. As a result $Q(x, y, t)$ will satisfy two more 3-wave NLEE

$$
\begin{aligned}
2\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} - \frac{a_1 - a_2}{c_1 - c_2}\frac{\partial u_1}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_1\epsilon_2 u_2^* u_3 = 0, \\
2\frac{\partial u_2}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_2}{\partial x} - \frac{a_1 - a_3}{c_1 - c_3}\frac{\partial u_2}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_1 u_1^* u_3 = 0, \\
2\frac{\partial u_3}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_3}{\partial x} - \frac{a_2 - a_3}{c_2 - c_3}\frac{\partial u_3}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_2 u_1^* u_2^* = 0.
\end{aligned}
$$

$$
\begin{aligned}
\kappa_1 &= a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2), \\
\kappa_2 &= a_1(c_2 - c_3) - a_2(c_1 - c_3) + a_3(c_1 - c_2).
\end{aligned}
$$

For N-wave equations related to Lie algebras $\mathfrak{g}$ of higher rank r we can add up to r auxiliary variables:

$$
r\frac{\partial Q}{\partial t} - \sum_{s=1}^{r}(\mathrm{ad}_{J_s}^{-1}\mathrm{ad}_J)\frac{\partial Q}{\partial x_s} - i\sum_{s=1}^{r}\mathrm{ad}_{J_s}^{-1}\,[[J, Q], [J_s, Q(\vec{x}, t)]] = 0
$$

where Q is an $n \times n$ off-diagonal matrix depending on $r+1$ variables. We remind that if $J = \operatorname{diag}(a_1, \ldots, a_n)$ then

$$(\mathrm{ad}\,_J Q)_{jk} \equiv ([J, Q])_{jk} = (a_j - a_k) Q_{jk}, \qquad (\mathrm{ad}\,_J^{-1} Q)_{jk} = \frac{1}{a_j - a_k} Q_{jk},$$

and similarly for the other J_s.

New 3-wave equations – $k \geq 2$

Let $\mathfrak{g} = sl(3)$ and

$$Q_1(\vec{x}, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad Q_2(\vec{x}, t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix},$$

Fix up $k = 2$. Then the Lax pair becomes

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]K, \xi^{\pm}(x,t,\lambda)] = 0,$$

where

$$U \equiv U_2 + \lambda U_1 = \left([J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)] \right) + \lambda [J, Q_1],$$
$$V \equiv V_2 + \lambda V_1 = \left([K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)] \right) + \lambda [K, Q_1].$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \mathrm{diag}\,(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus Q_1 and Q_2 get reduced into:

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix},$$

New type of integrable 3-wave equations:

$$i(a_1-a_2)\frac{\partial u_1}{\partial t}-i(b_1-b_2)\frac{\partial u_1}{\partial x}+\epsilon\kappa u_2^* u_3+\epsilon\frac{\kappa(a_1-a_2)}{(a_1-a_3)}u_1|u_2|^2=0,$$

$$i(a_2-a_3)\frac{\partial u_2}{\partial t}-i(b_2-b_3)\frac{\partial u_2}{\partial x}+\epsilon\kappa u_1^* u_3-\epsilon\frac{\kappa(a_2-a_3)}{(a_1-a_3)}|u_1|^2u_2=0,$$

$$i(a_1-a_3)\frac{\partial u_3}{\partial t}-i(b_1-b_3)\frac{\partial u_3}{\partial x}-\frac{i\kappa}{a_1-a_3}\frac{\partial(u_1u_2)}{\partial x}$$
$$+\epsilon\kappa\left(\frac{a_1-a_2}{a_1-a_3}|u_1|^2+\frac{a_2-a_3}{a_1-a_3}|u_2|^2\right)u_1u_2+\epsilon\kappa u_3(|u_1|^2-|u_2|^2)=0,$$

where

$$\kappa=a_1(b_2-b_3)-a_2(b_1-b_3)+a_3(b_1-b_2),\qquad u_3=w_3+\frac{2a_2-a_1-a_3}{2(a_1-a_3)}u_1u_2.$$

The diagonal terms in the Lax representation are λ-independent.

Two of them read:

$$
\begin{aligned}
i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,\\
i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,
\end{aligned}
$$

These relations are satisfied identically as a consequence of the NLEE.

New types of 4-wave interactions

The Lax pair for these new equations will be provided by:

$$
\begin{aligned}
L\psi = i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,\\
M\psi = i\frac{\partial \psi}{\partial t} + (V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K)\psi(x,t,\lambda) = 0,
\end{aligned}
$$

where $U_j(x,t)$ and $V_j(x,t)$ are fast decaying smooth functions taking values in the Lie algebra $so(5)$

$$U_1(x,t) = [J, Q_1(x,t)], \qquad U_2(x,t) = [J, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}\,^2_{Q_1} J,$$
$$V_1(x,t) = [K, Q_1(x,t)], \qquad V_2(x,t) = [K, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}\,^2_{Q_1} K.$$

Here $\mathrm{ad}\,_{Q_1} X \equiv [Q_1(x,t), X]$.

Assume $Q_1(x,t)$ and $Q_2(x,t)$ to be generic elements of $so(5)$:

$$Q_1(x,t) = \sum_{\alpha\in\Delta_+} (q^1_\alpha E_\alpha + p^1_\alpha E_{-\alpha}) + r^1_1 H_{e_1} + r^1_2 H_{e_2},$$

$$Q_2(x,t) = \sum_{\alpha\in\Delta_+} (q^2_\alpha E_\alpha + p^2_\alpha E_{-\alpha}) + r^2_1 H_{e_1} + r^2_2 H_{e_2},$$

$$J = a_1 H_{e_1} + a_2 H_{e_2} = \mathrm{diag}\,(a_1, a_2, 0, -a_2, -a_1),$$
$$K = b_1 H_{e_1} + b_2 H_{e_2} = \mathrm{diag}\,(b_1, b_2, 0, -b_2, -b_1),$$

Next we impose on $Q_1(x,t)$ and $Q_2(x,t)$ the natural reduction

$$B_0 U(x,t,\epsilon\lambda^*)^\dagger B_0^{-1} = U(x,t,\lambda), \qquad B_0 = \mathrm{diag}\,(1,\epsilon,1,\epsilon,1), \quad \epsilon^2 = 1.$$

As a result:

$$B_0(\chi^+(x,t,\epsilon\lambda^*))^\dagger B_0^{-1} = (\chi^-(x,t,\lambda))^{-1}, \qquad B_0(T(t,\epsilon\lambda^*))^\dagger B_0^{-1} = (T(t,\lambda))^{-1},$$

which provide $p^1_\alpha = \epsilon(q^1_\alpha)^*$, $p^2_\alpha = \epsilon(q^2_\alpha)^*$. Then the Lax representation will be a (rather complicated) system of 8 NLEE for the 8 independent matrix elements q^1_α and q^2_α.

However we can impose additional $\mathbb{Z}_2$ reduction condition

$$D\xi^\pm(x,t,-\lambda)\hat{D} = \xi^\pm(x,t,\lambda), \quad DQ(x,t,-\lambda)\hat{D} = Q(x,t,\lambda), \; D = \mathrm{diag}\,(1,-1,1,-1,1)$$

$$
\begin{aligned}
Q_1(x,t) &= u_1 E_{e_1-e_2} + u_2 E_{e_2} + u_3 E_{e_1+e_2} + v_1 E_{-e_1+e_2} + v_2 E_{-e_2} + v_3 E_{-e_1-e_2} \\
&= \begin{pmatrix} 0 & u_1 & 0 & u_3 & 0 \\ v_1 & 0 & u_2 & 0 & u_3 \\ 0 & v_2 & 0 & u_2 & 0 \\ v_3 & 0 & v_2 & 0 & u_1 \\ 0 & v_3 & 0 & v_1 & 0 \end{pmatrix}, \\
Q_2(x,t) &= u_4 E_{e_1} + v_4 E_{-e_1} + w_1 H_{e_1} + w_2 H_{e_2} \\
&= \begin{pmatrix} w_1 & 0 & u_4 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ w_4 & 0 & 0 & 0 & u_4 \\ 0 & 0 & 0 & -w_2 & 0 \\ 0 & 0 & -v_4 & 0 & -w_1 \end{pmatrix}, \\
J &= a_1 H_{e_1} + a_2 H_{e_2} = \operatorname{diag}(a_1, a_2, 0, -a_2, -a_1), \\
K &= b_1 H_{e_1} + b_2 H_{e_2} = \operatorname{diag}(b_1, b_2, 0, -b_2, -b_1),
\end{aligned}
$$

Combining both reductions for the matrix elements of $Q_j(x,t)$ we have:

$$
v_1 = \epsilon u_1^*, \qquad v_2 = \epsilon u_2^*, \qquad v_3 = \epsilon u_3^*, \qquad v_4 = u_4^*,
$$

The commutativity condition for the Lax pair

$$i\left(\frac{\partial V_2}{\partial x} + \lambda\frac{\partial V_1}{\partial x}\right) - i\left(\frac{\partial U_2}{\partial t} + \lambda\frac{\partial U_1}{\partial t}\right) + [U_2 + \lambda U_1 - \lambda^2 J, V_2 + \lambda V_1 - \lambda^2 K] = 0$$

must hold identically with respect to λ. The terms proportional to λ^4, λ^3 and λ^2 vanish identically. The term proportional to λ and the λ-independent term vanish provided Q_i satisfy the NLEE:

$$\begin{aligned} &i\frac{\partial V_1}{\partial x} - i\frac{\partial U_1}{\partial t} + [U_2, V_1] + [U_1, V_1] = 0, \\ &i\frac{\partial V_2}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2, V_2] = 0. \end{aligned}$$

In components the corresponding NLEE:

$$
\begin{aligned}
&- 2i(a_1 - a_2)\frac{\partial u_1}{\partial t} + 2i(b_1 - b_2)\frac{\partial u_1}{\partial x} + \kappa\epsilon u_2^*(\epsilon u_2^* u_3 - u_1u_2 - 2u_4) = 0,\\
&- 2ia_2\frac{\partial u_2}{\partial t} + 2ib_2\frac{\partial u_2}{\partial x} - \kappa(u_2\epsilon(|u_3|^2 - |u_1|^2) + 2u_3u_4^* + 2\epsilon u_1^* u_4) = 0,\\
&- 2i(a_1 + a_2)\frac{\partial u_3}{\partial t} + 2i(b_1 + b_2)\frac{\partial u_3}{\partial x} + \kappa u_2(\epsilon u_2^* u_3 - u_1u_2 + 2u_4) = 0,\\
&- 2ia_1\frac{\partial u_4}{\partial t} + 2ib_1\frac{\partial u_4}{\partial x} + i\frac{\partial}{\partial t}(-(2a_2 - a_1)u_1u_2 + (2a_2 + a_1)\epsilon u_2^* u_3)\\
&+ i(2b_2 - b_1)\frac{\partial(u_1u_2)}{\partial x} - i(2b_2 + b_1)\epsilon\frac{\partial(u_2^* u_3)}{\partial x} - \kappa\left(2\epsilon u_4(|u_1|^2 - |u_3|^2)\right.\\
&\left.+ \epsilon u_1u_2(|u_1|^2 + 3|u_3|^2) - u_3u_2^*(3|u_1|^2 + |u_3|^2)\right) = 0.
\end{aligned}
$$

Let us now introduce

$$
U_4 = u_4 - \frac{1}{2a_1}((a_1 - a_2)u_1u_2 + (a_1 + a_2)\epsilon u_3u_2^*).
$$

As a result we get:

$$
\begin{aligned}
&-2i(a_1-a_2)\frac{\partial u_1}{\partial t}+2i(b_1-b_2)\frac{\partial u_1}{\partial x}-\frac{\kappa\epsilon}{a_1}u_2^*(2a_1U_4+\epsilon a_2u_2^*u_3+(2a_1-a_2)u_1u_2)=0,\\
&-2ia_2\frac{\partial u_2}{\partial t}+2ib_2\frac{\partial u_2}{\partial x}-\frac{\kappa\epsilon}{a_1}u_2((2a_1+a_2)|u_3|^2-a_2|u_1|^2)\\
&\qquad-2\kappa(u_3U_4^*+\epsilon u_1^*U_4+u_1^*u_2^*u_3)=0,\\
&-2i(a_1+a_2)\frac{\partial u_3}{\partial t}+2i(b_1+b_2)\frac{\partial u_3}{\partial x}+\frac{\kappa}{a_1}u_2(\epsilon(2a_1+a_2)u_2^*u_3-a_2u_1u_2+2a_1U_4)=0,\\
&-2ia_1\frac{\partial U_4}{\partial t}+2ib_1\frac{\partial U_4}{\partial x}+\frac{i\kappa}{a_1}\frac{\partial u_1u_2}{\partial x}-\frac{i\kappa\epsilon}{a_1}\frac{\partial u_2^*u_3}{\partial x}\\
&-\frac{\kappa}{a_1}\left(2\epsilon U_4(|u_1|^2-|u_3|^2)+(\epsilon u_1u_2-u_3u_2^*)((2a_1-a_2)|u_1|^2+(2a_1+a_2)|u_3|^2)\right)=0,
\end{aligned}
$$

More new equations

Consider $\mathfrak{g} \simeq sl(3)$; $k = 3$

$$
\begin{aligned}
L\psi &\equiv i\frac{\partial \psi}{\partial x} + (U_3 + \lambda U_2 + \lambda^2 U_1 - \lambda^3 J)\psi(x,t,\lambda) = 0, \\
M\psi &\equiv i\frac{\partial \psi}{\partial t} + (V_3 + \lambda V_2 + \lambda^2 V_1 - \lambda^3 K)\psi(x,t,\lambda) = 0,
\end{aligned}
$$

The NLEE

$$
i\frac{\partial U}{\partial t} - i\frac{\partial V}{\partial x} - [U(x,t,\lambda), V(x,t,\lambda)] = 0.
$$

Independent matrices: Q_1, Q_2, Q_3. Reduction: $Q_j = \epsilon Q_j^*$;
Number of independent functions: 9 or less if reductions are used.

Mikhailov reductions: Use automorphism of third order:
Choose $Z = \operatorname{diag}(1, \omega, \omega^2)$ with $\omega^3 = 1$.

$$J = \mathrm{diag}\,(a_1, a_2, a_3), \qquad K = \mathrm{diag}\,(b_1, b_2, b_3),$$

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ 0 & 0 & u_2 \\ u_3 & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & v_3 \\ v_1 & 0 & 0 \\ 0 & v_2 & 0 \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} d_3 & 0 & 0 \\ 0 & d_4 & 0 \\ 0 & 0 & d_5 \end{pmatrix}$$

$$U_1 = \mathrm{ad}_{\,Q_1} J, \qquad U_2 = \mathrm{ad}_{\,Q_2} J + \frac{1}{2}\mathrm{ad}_{\,Q_1}\mathrm{ad}_{\,Q_1} J,$$

$$U_3 = \mathrm{ad}_{\,Q_3} J + \frac{1}{2}(\mathrm{ad}_{\,Q_1}\mathrm{ad}_{\,Q_2} + \mathrm{ad}_{\,Q_2}\mathrm{ad}_{\,Q_1})J + \frac{1}{6}\mathrm{ad}^{\,3}_{\,Q_1} J,$$

$$V_1 = \mathrm{ad}_{\,Q_1} K, \qquad V_2 = \mathrm{ad}_{\,Q_2} K + \frac{1}{2}\mathrm{ad}_{\,Q_1}\mathrm{ad}_{\,Q_1} K,$$

$$V_3 = \mathrm{ad}_{\,Q_3} K + \frac{1}{2}(\mathrm{ad}_{\,Q_1}\mathrm{ad}_{\,Q_2} + \mathrm{ad}_{\,Q_2}\mathrm{ad}_{\,Q_1})K + \frac{1}{6}\mathrm{ad}^{\,3}_{\,Q_1} K,$$

$$U(x,t,\lambda) = U_3 + \lambda U_2 + \lambda^2 U_1 + \lambda^3 J, \qquad V(x,t,\lambda) = V_3 + \lambda V_2 + \lambda^2 V_1 + \lambda^3 K,$$

$$L \equiv i\frac{\partial}{\partial x} + U(x,t,\lambda), \qquad M \equiv i\frac{\partial}{\partial t} + V(x,t,\lambda),$$

NLEE:

$$i\frac{\partial V}{\partial x} - i\frac{\partial U}{\partial t} + [U(x,t,\lambda), V(x,t,\lambda)] = 0.$$

$$i(a_1-a_2)\frac{\partial u_1}{\partial t} - i(b_1-b_2)\frac{\partial u_1}{\partial x} + \frac{\kappa}{4}(u_1u_2+2v_3)(u_1u_3-2v_2) = 0,$$
$$i(a_2-a_3)\frac{\partial u_2}{\partial t} - i(b_2-b_3)\frac{\partial u_2}{\partial x} + \frac{\kappa}{4}(u_2u_3+2v_1)(u_1u_2-2v_3) = 0,$$
$$i(a_1-a_3)\frac{\partial u_3}{\partial t} - i(b_1-b_3)\frac{\partial u_3}{\partial x} - \frac{\kappa}{4}(u_2u_3-2v_1)(u_1u_3+2v_2) = 0,$$

$$
\begin{aligned}
&- i(a_1 - a_2)\frac{\partial v_1}{\partial t} + i(b_1 - b_2)\frac{\partial v_1}{\partial x} + \frac{ia_3^\vee}{2}\frac{\partial(u_2u_3)}{\partial t} - \frac{ib_3^\vee}{2}\frac{\partial(u_2u_3)}{\partial x} \\
&\qquad - \frac{\kappa}{2}\left(u_3v_3(u_2u_3 - 2v_1) + u_2u_3(u_2v_2 + 2u_1v_1) + 2u_2v_1v_2\right) = 0, \\
&- i(a_2 - a_3)\frac{\partial v_2}{\partial t} + i(b_2 - b_3)\frac{\partial v_2}{\partial x} + \frac{ia_1^\vee}{2}\frac{\partial(u_1u_3)}{\partial t} - \frac{ib_1^\vee}{2}\frac{\partial(u_1u_3)}{\partial x} \\
&\qquad - \frac{\kappa}{2}\left(u_3v_3(u_1u_3 + 2v_2) + u_1u_3(u_1v_1 + 2u_2v_2) - 2u_1v_1v_2\right) = 0, \\
&i(a_1 - a_3)\frac{\partial v_3}{\partial t} - i(b_1 - b_3)\frac{\partial v_3}{\partial x} + \frac{ia_2^\vee}{2}\frac{\partial(u_1u_2)}{\partial t} - \frac{ib_2^\vee}{2}\frac{\partial(u_1u_2)}{\partial x} \\
&\qquad - \frac{\kappa}{2}\left(u_1v_2(u_1v_1 + 2u_3v_3) + u_2v_2(u_1u_2 - 2v_3) + 2u_1v_1v_3\right) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1^\vee &= 2a_1 - a_2 - a_3, \qquad & a_2^\vee &= 2a_2 - a_1 - a_3, \qquad & a_3^\vee &= 2a_3 - a_1 - a_2, \\
b_1^\vee &= 2b_1 - b_2 - b_3, \qquad & b_2^\vee &= 2b_2 - b_1 - a_3, \qquad & a_3^\vee &= 2a_3 - a_1 - a_2,
\end{aligned}
$$

Soliton solutions

Dressing method – Zakharov and Shabat 1974. Developed for $k = 1$, but it can be extended to $k > 1$!
Let $\xi_0^+(x,t,\lambda)$ be a regular solution to the RHP

$$\xi_0^+(x,t,\lambda) = \xi_0^-(x,t,\lambda)G_0(x,t,\lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^+(x,t,\lambda) = \mathbb{1},$$

$$i\frac{\partial G_0}{\partial x} - \lambda^k[J, G_0(x,t,\lambda)] = 0, \qquad i\frac{\partial G_0}{\partial t} - \lambda^k[K, G_0(x,t,\lambda)] = 0.$$

where $J, K \in \mathfrak{h} \subset \mathfrak{g}$.

Construct new singular solution to the RHP that will depend on additional parameters:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad G(x,t,\lambda) = u_- G_0(x,t,\lambda)u_-^{-1},$$

$$\xi^+(x,t,\lambda) = u(x,t,\lambda)\xi_0^+(x,t,\lambda)u_-^{-1}(\lambda), \qquad u_- = \lim_{x\to-\infty} u(x,t,\lambda).$$

Construction of $u(x,t,\lambda)$ by Zakharov-Shabat anzatz:

$$u(x,t,\lambda) = \mathbb{1} + \sum_{s=1}^{k} \frac{P_s(x,t)}{\lambda - \lambda_s}, \qquad \lambda_s = \lambda_1 \omega^{s-1}, \qquad \omega = e^{2\pi i/k}.$$

Typical form of the residues:

$$\begin{aligned} P_s &= \frac{|n_s\rangle\langle m_s|}{\langle m_s|n_s\rangle}, \\ |n_s\rangle &= \xi_0^+(x,t,\lambda_s) e^{-i\lambda_s^k Jx - i\lambda_s^k Kt} |n_{0,s}\rangle, \\ \langle m_s| &= \langle m_{0,s}| e^{i\lambda_s^k Jx + i\lambda_s^k Kt} \hat{\xi}_0^-(x,t,\lambda_s), \end{aligned}$$

The effect of the dressing: adds new discrete eigenvalues to the spectrum of L!

BD.I-type MNLS - soliton interactions

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger, \vec{q})\vec{q}(x,t) - (\vec{q}, s_0\vec{q})s_0\vec{q}^*(x,t) = 0, \qquad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} L\psi(x,t,\lambda) &\equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \\ M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\ V_1(x,t) &= Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}\,_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}\,_J^{-1}Q, Q(x,t)\right]. \end{aligned}$$

where $J = \mathrm{diag}(1, 0, \ldots 0, -1)$ and

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \quad S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

Here $(E_{kn})_{ij} = \delta_{ik}\delta_{nj}$ and the $2r-1$-vectors $\vec{q}$ and $\vec{p} = \vec{q}^*$ take the form

$$\vec{q} = (q_1, \ldots, q_{r-1}, q_0, q_{-1}, \ldots, q_{-r+1})^T,$$

The dressing factor $u(x,\lambda) \in SO(2r+1)$

$$u(x,\lambda) = \mathbb{1} + (c(\lambda)-1)P(x,t) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x,t), \qquad \overline{P} = S_0^{-1}P^T S_0,$$

where $P(x,t)$ and $\overline{P}(x,t)$ are projectors which satisfy $P\overline{P}(x,t) = 0$.

$$P(x,t) = \frac{|n_1(x,t)\rangle\langle n_1^\dagger(x,t)|}{\langle n_1^\dagger(x,t)|n_1(x,t)\rangle},$$

$$|n_1(x,t)\rangle = \chi_0^+(x,t,\lambda_1^+)|n_{0,1}\rangle, \quad c(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \quad \langle n_{0,1}|S_0|n_{0,1}\rangle = 0.$$

Taking the limit $\lambda \to \infty$ we get that

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^- - \lambda_1^+)[J, P(x,t) - \overline{P}(x,t)].$$

If $Q_{(0)} = 0$ and put $\lambda_1^\pm = \mu \pm i\nu$, $\chi_0^+(x,t,\lambda) = e^{-i\lambda Jx}$:

$$q_k^{(1\mathrm{s})}(x,t) = -2i\nu\left(P_{1k}(x,t) + (-1)^k P_{\bar{k},2r+1}(x,t)\right),$$

where $\bar{k} = 2r+2-k$.

The one-soliton solution reads

$$
\begin{aligned}
q_k &= \frac{-i\nu e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2}\left(\alpha_k e^{z-i\phi_k} + (-1)^k \alpha_{\bar{k}} e^{-z+i\phi_{\bar{k}}}\right), \\
v &= \frac{\nu^2-\mu^2}{\mu}, \qquad u = -2\mu, \qquad z(x,t) = \nu(x-ut-\xi_0), \qquad\qquad (1) \\
\xi_0 &= \frac{1}{2\nu}\ln\frac{|n_{0,2r+1}|}{|n_{0,1}|}, \qquad \alpha_k = \frac{|n_{0,k}|}{\sqrt{|n_{0,1}||n_{0,2r+1}|}}, \qquad \Delta_0^2 = \frac{\sum_{k=2}^{2r}|n_{0,k}|^2}{2|n_{0,1}n_{0,2r+1}|},
\end{aligned}
$$

and $\delta_0 = \arg n_{0,1}/\mu = -\arg n_{0,2r+1}/\mu$, $\phi_k = \arg n_{0,k}$ and

$$
\sum_{k=1}^{r} 2(-1)^{k+1} n_{0,k} n_{0,\bar{k}} + (-1)^r n_{0,r+1}^2 = 0.
$$

For $r=2$ we identify $\Phi_1 = q_2$, $\Phi_0 = q_3/\sqrt{2}$ and $\Phi_3 = q_4$ and get:

$$
\begin{aligned}
\Phi_{\pm 1} &= -\frac{2i\nu\sqrt{\alpha_2\alpha_4}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 1}\cosh z_{\pm 1} - i\sin\phi_{\pm 1}\sinh z_{\pm 1}\right), \\
\delta_{\pm 1} &= \delta_0 \mp \frac{\phi_2-\phi_4}{2\mu}, \qquad \phi_{\pm 1} = \frac{\phi_2+\phi_4}{2} \qquad z_{\pm 1} = z \mp \frac{1}{2}\ln\frac{\alpha_4}{\alpha_2},
\end{aligned}
$$

$$\Phi_0 = -\frac{\sqrt{2} i\nu \alpha_3 e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} \left(\cos\phi_3 \sinh z - i \sin\phi_3 \cosh z \right).$$

N-soliton interaction for the BD.I VNLS

Apply the dressing method in the form:

$$\begin{aligned}
\chi^{\pm}_{(1)}(x,\lambda) &= u_1(x,t,\lambda)\chi^{\pm}_{(0)}(x,t,\lambda)\hat{u}_1^{-}(\lambda), \\
u_1(x,\lambda) &= \mathbb{1} + (c_1(\lambda)-1)P_1(x,t) + (c_1^{-1}(\lambda)-1)\bar{P}_1(x,t), \\
c_1(\lambda) &= \frac{\lambda-\lambda_1^+}{\lambda-\lambda_1^-}, \qquad P_1 = \frac{|n_1\rangle\langle n_1|}{\langle n_1|n_1\rangle}, \\
|n_1\rangle &= \chi^{+}_{(0)}(x,t,\lambda_1^+)|n_{10}\rangle, \qquad \langle n_1| = \langle n_{10}|\hat{\chi}^{-}_{(0)}(x,t,\lambda_1^-).
\end{aligned}$$

Then the FAS $\chi^{\pm}_{(0)}(x,t,\lambda)$ correspond to the dressed Lax operator $L_{(1)}$ with potential $Q_{(1)}(x,t)$ which has two discrete eigenvalues: $\lambda_1^{\pm} \in \mathbb{C}_{\pm}$.

$$Q_{(1)} = Q_{(0)} - 2i\nu_1[J, P_1(x,t) - \bar{P}_1(x,t)].$$

Choose $Q_{(0)} = 0$; therefore

$$\chi^{\pm}_{(0)}(x,t,\lambda) = e^{-i(\lambda x + \lambda^2 t)J}.$$

N-soliton solution by applying N times the above procedure:

$$u_{\rm Ns}(x,\lambda) = u_N(x,t,\lambda)\cdots u_1(x,t,\lambda)\hat{u}_N^-(\lambda)\cdots\hat{u}_1^-(\lambda),$$

$$\begin{aligned}
u_s(x,\lambda) &= \mathbb{1} + (c_s(\lambda)-1)P_s(x,t) + (c_s^{-1}(\lambda)-1)\bar{P}_s(x,t),\\
c_s(\lambda) &= \frac{\lambda-\lambda_s^+}{\lambda-\lambda_s^-}, \qquad P_s = \frac{|\boldsymbol{n}_s\rangle\langle\boldsymbol{n}_s|}{\langle\boldsymbol{n}|\boldsymbol{n}\rangle},\\
|\boldsymbol{n}_s\rangle &= u_{s-1}(x,\lambda_s^+)\cdots u_1(x,\lambda_s^+)\hat{u}_1^-(\lambda_s^+)\cdots\hat{u}_{s-1}^-(\lambda_s^+)|n_s\rangle,\\
\langle\boldsymbol{n}_s| &= \langle n_s|u_{s-1}^-(\lambda_s^-)\cdots u_1^-(\lambda_s^-)\hat{u}_1(x,\lambda_s^-)\cdots\hat{u}_{s-1}(x,\lambda_s^-),\\
|n_s\rangle &= \chi^+_{(0)}(x,t,\lambda_s^+)|n_{s0}\rangle = e^{(z_s-i\phi_s)J}|n_{s0}\rangle,\\
\langle n_s| &= \langle n_{s0}|\hat{\chi}^-_{(0)}(x,t,\lambda_s^-) = \langle n_{s0}|e^{(z_s+i\phi_s)J},\\
z_s &= \nu_s(x+2\mu_s t), \qquad \phi_s = \mu_s x + (\mu_s^2-\nu_s^2)t.
\end{aligned}$$

The asymptotics of $P_s(x,t)$ for $z_s \to \pm\infty$ are given by:

$$\lim_{z_s\to\infty} P_s(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \lim_{z_s\to-\infty} P_s(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

The asymptotics of $u_s(x,t,\lambda)$ for $z_s \to \pm\infty$ are given by:

$$\lim_{z_s\to\infty} u_s(x,t,\lambda) \equiv u_s^+(\lambda) = e^{J\ln c_s(\lambda)}, \qquad \lim_{z_s\to-\infty} u_s(x,t,\lambda) \equiv u_s^-(\lambda) = e^{-J\ln c_s(\lambda)}.$$

Derive the asymptotics of the N-soliton dressing factor $u_{\rm Ns}(x,t)$ keeping z_N fixed and taking $t \to \infty$ and $t \to -\infty$. In doing this we assume that each two solitons move with different speeds, i.e.:

$$\mu_k \neq \mu_s, \qquad \text{for} \qquad k \neq s.$$

We start with the asymptotics of $u_s(x,\lambda)$ and for $\mu_s > \mu_N$ we easily get the result:

$$\lim_{\substack{t\to\infty \\ z_N = \text{fix}}} P_s(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lim_{\substack{t\to-\infty \\ z_N = \text{fix}}} P_s(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

If $\mu_s < \mu_N$ we get:

$$\lim_{\substack{t\to\infty\\ z_N=\text{fix}}} P_s(x,t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \lim_{\substack{t\to-\infty\\ z_N=\text{fix}}} P_s(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lim_{\substack{t\to\infty\\ z_N=\text{fix}}} u_s(x,t) = \begin{pmatrix} c_s(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & c_s^{-1}(\lambda) \end{pmatrix} = e^{J\ln c_1(\lambda)},$$

$$\lim_{\substack{t\to-\infty\\ z_N=\text{fix}}} u_s(x,t) = \begin{pmatrix} c_s^{-1}(\lambda) & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & c_s(\lambda) \end{pmatrix} = e^{-J\ln c_1(\lambda)},$$

Finally we evaluate the limits of $P_N(x,t)$ with the result:

$$\lim_{\substack{t\to\infty \\ z_N=\mathrm{fix}}} P_N(x,t) = \frac{|\boldsymbol{n}_N^+\rangle\langle n_N^+|}{\langle n_N^+|n_N^+\rangle}, \qquad \lim_{\substack{t\to-\infty \\ z_N=\mathrm{fix}}} P_s(x,t) = \frac{|\boldsymbol{n}_N^-\rangle\langle n_N^-|}{\langle n_N^-|n_N^-\rangle},$$

$$|\boldsymbol{n}_N^+\rangle = \prod_{s\in\sigma_\mu^+} (u^+(\lambda_N^+))^2|n_N\rangle, \qquad |\boldsymbol{n}_N^-\rangle = \prod_{s\in\sigma_\mu^-} (u^+(\lambda_N^+))^2|n_N\rangle,$$

$$\sigma_\mu^+ \equiv \{\mu_s,\ \mu_s > \mu_N\}_{s=1}^{N-1}, \qquad \sigma_\mu^- \equiv \{\mu_s,\ \mu_s < \mu_N\}_{s=1}^{N-1},$$

Thus we find:

$$\lim_{\substack{t\to\infty \\ z_N=\mathrm{fix}}} Q_{\mathrm{Ns}}(x,t) = Q_{1\mathrm{s}}(z_N^+,\phi_N^+), \qquad \lim_{\substack{t\to-\infty \\ z_N=\mathrm{fix}}} Q_{\mathrm{Ns}}(x,t) = Q_{1\mathrm{s}}(z_N^-,\phi_N^-),$$

$$z_N^+ = z_N + 2\sum_{s\in\sigma_\mu^+} \ln\left|\frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-}\right|, \qquad z_N^- = z_N - 2\sum_{s\in\sigma_\mu^-} \ln\left|\frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-}\right|$$

$$\phi_N^+ = \phi_N + 2\sum_{s\in\sigma_\mu^+} \arg\frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-}, \qquad \phi_N^- = \phi_N - 2\sum_{s\in\sigma_\mu^+} \arg\frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-}.$$

Thus the N-soliton interactions of the **BD.I**-VNLS are like for the scalar NLS case. The only effect of the interaction is the shifts of relative center-of-mass coordinates and of the phases:

$$
\begin{aligned}
z_N^+ - z_N^- &= 2 \sum_{s\in\sigma_\mu^+} \ln \left| \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} \right| - 2 \sum_{s\in\sigma_\mu^-} \ln \left| \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} \right|, \\
\phi_N^+ - \phi_N^- &= 2 \sum_{s\in\sigma_\mu^+} \arg \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-} - 2 \sum_{s\in\sigma_\mu^-} \arg \frac{\lambda_N^+ - \lambda_s^+}{\lambda_N^+ - \lambda_s^-},
\end{aligned}
$$

Conclusions and open questions

- More classes of new integrable equations: i) higher rank simple Lie algebras; ii) different types of grading; iii) different power k of the polynomials $U(\vec{x}, t, \lambda)$ and $V(\vec{x}, t, \lambda)$ and iv) different reductions of U and V.

- These new NLEE must be Hamiltonian. View the jets $U(\vec{x}, t, \lambda)$ and $V(\vec{x}, t, \lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.

- Apply Zakharov-Shabat dressing method for constructing their N-soliton solutions and study their interactions.