Symmetries and invariant solutions for evolutionary equations

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I.1. Integrability of the dynamical systems

- Dynamical systems are usually described through nonlinear differential equations.

  If solutions exist, the diffential system is said to be *integrable*.
- There is no a general theory/procedure allowing to solve completely nonlinear PDEs.
- Sometimes it is quite enough to decide if the system is integrable or not. *Methods*:

  Hirota' s bilinear method

  Backlund transformation

  Inverse scattering

  Lax pair operator

  Painleve analysis,

  Symmetry approach, etc.
- *The symmetry method* - efficient techniques in studying integrability. It allows to obtain:
  - The first integrals/invariants specific for the symmetry transformations;
  - Classes of exact solutions through *similarity reduction* (reduction of PDEs to ODEs).
  - New solutions starting from known ones.

I.2. Point-like symmetries. Lie operators.

- Let us consider a system of q partial differential equations (PDEs):

$$\Delta = \{\Delta^{\nu}(t,x,u(x,t),u^{(n)}(x,t))\}_{\nu=1}^{q} = 0 \qquad (1.1)$$

defined on a domain $M \subset R^{p}$ (i.e. a connected open subset of R^{p}) with at most $n-$ th order partial derivatives of $u(x,t)=\{u^{1}(x,t),...,u^{q}(x,t)\}$ in the space-time variables $(x,t)=\{t,x^{1},...,x^{p}\}$. The notation $u^{\alpha(J)}(x,t)$ designates the partial derivatives of $\{u^{\alpha}(x,t),\alpha=1,...q\}$ up to the J- th order:

$$u^{\alpha(J)} = \frac{\partial^{J} u^{\alpha}}{\partial t^{j_0}\partial x^{1(j_1)}\partial x^{2(j_2)}...\partial x^{p(j_p)}} \equiv D^{J}u^{\alpha};\ J = j_0 + j_1 + ... + j_p \qquad (1.2)$$

- A *point-like transformation* may be defined through an infinitesimal parameter **ε** by:

$$\begin{aligned} &t' = t + \delta t,\ \delta t = \varepsilon\varphi(x,t) + O(\varepsilon^{2}) + ... \\ &x = \{x^{i}, i = 1,..., p\}; x' = \{x'^{i}, i = 1,..., p\} \\ &x'^{i} = x^{i} + \delta x^{i},\ \delta x^{i} = \varepsilon\cdot\xi^{i}(x,t) + O(\varepsilon^{2}) + ... \end{aligned} \qquad (1.3)$$

- The transformations (1.3) induce a first order variation of the dependent variables given by:

$$\delta u = u'(x',t') - u(x,t) = \frac{\partial u}{\partial t}\delta t + \sum_{i=1}^{p}\frac{\partial u}{\partial x_i}\delta x_i \equiv \varepsilon \cdot X \cdot u(x,t) \qquad (1.4)$$

- The operator X denotes the generator of the infinitesimal point-like transformations and is called *Lie operator*. In the first order approximation its concrete form is:

$$X = \varphi\frac{\partial}{\partial t} + \sum_{i=1}^{p}\xi^i(t,x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\phi^\alpha(t,x,u)\frac{\partial}{\partial u^\alpha} \qquad (1.5)$$

- Let us denote by $X^{(n)}$ the n-th order extension of the Lie infinitesimal symmetry operator:

$$X^{(n)} = X + \sum_{\alpha=1}^{q}\sum_{J}\phi^{\alpha(J)}(x,u^{(n)})\frac{\partial}{\partial u^{\alpha(J)}}$$

$$\phi^{\alpha(J)}(x,u^{(n)}) = D^J[\phi^\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha] + \sum_{i=1}^{p}\xi^i u_i^{\alpha(J)}, \quad u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \alpha = 1,\dots q \qquad (1.6)$$

- Lie symmetry method requires to impose the following invariance condition:

[P.J.Olver- Applications of Lie Groups to Differential Equations, GTM 107, Second edition, Springer-Verlag, 1993]

$$\Delta' \equiv X^{(n)}(\Delta)\big|_{\Delta=0} = 0 \;\; for\, \Delta \equiv \{\Delta^\nu,\; \nu = 1,\dots q\} \qquad (1.7)$$

Within (1.7) the equation (1.1) could change its form but not the class of solutions.

- **CONCLUSION:** For each PDE there is a local group of transformations on the space of its independent and dependent variables called symmetry group that maps the set of all analytical solutions on itself. Knowledge of Lie symmetries allows to construct group-invariant solutions.

I.3. Invariants and similarity reduction

- One of the advantages of the method: find solutions of the original PDEs by solving ODEs. These ODEs, called *reduced equations*, are obtained by introducing suitable new variables, determined as invariant functions in respect to the Lie generators.
- By applying Lie operators on the equations, one gets the *determining system*. It allows to effectively find the symmetry generators $\{\varphi(t,x,u), \xi^i(t,x,u), \phi^\alpha(t,x,u)\}$
- Knowing the symmetry generators we have to solve the associated *characteristic equations*:

$$\frac{dt}{\varphi} = \frac{dx^1}{\xi^1} = \ldots = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = \ldots = \frac{du^q}{\phi_q} \tag{1.8}$$

- By integrating, we obtain the invariants of the analyzed system $\{I_r, r = 1,\ldots, p+q\}$.
- *Similarity reduction*: the invariants are chosen as *similarity variables* and they are expressed in terms of the original ones: $p+1$ independent variables and q dependent. We get a set of differential equations with only $(p+q)$ variables.

I.3.1 Generalizations *of the Lie symmetry method*

1. The *non-classical symmetry method* (Bluman and Cole): added the invariance surface condition:

$$Q^{\alpha}(x,u^{(1)}) \equiv \phi_{\alpha}(x,u) - \sum_{i=1}^{p} \xi^{i}(x,u)\frac{\partial u^{\alpha}}{\partial x^{i}} = 0,\ \alpha = \overline{1,q} \tag{1.9}$$

Consequences:

- Smaller number of determining equations for the infinitesimals $\xi^{i}(x,u),\ \phi_{\alpha}(x,u)$.
- More solutions than the CSM (any classical symmetry is a non-classical one)

2. The *direct method* (Clarkson and Kruskal): a direct, algorithmic method for finding symmetry reductions.
3. The *differential constraint approach* (Olver and Rosenau): the original system of partial differential equations can be enlarged by appending additional differential constraints (side conditions), resulting an over-determined system of partial differential equations.
4. The *generalized conditional symmetries method* or *conditional Lie-Bäcklund symmetries* (Fokas, Liu and Zhdanov).

I.4.The inverse symmetry problem

- *The* ***direct symmetry problem*** for evolutionary equations consists in:
  - Determining the Lie symmetry group corresponding to a given evolutionary equation.
  - Obtaining the invariants associated to each symmetry operator.
  - Obtaining some reduced equations with the similarity reduction procedure.
  - Solving the reduced equation and generating similarity solutions of the model.
- The ***inverse symmetry problem***: what is the largest class of evolutionary equations which are equivalent from the point of view of their symmetries?
- **Example** of a $2D$ dynamical system:

$$u_t = A(x,y,t,u)u_{xy} + B(x,y,t,u)u_x u_y + C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} + \\ + E(x,y,t,u)u_y + F(x,y,t,u)u_x + G(x,y,t,u) \qquad (1.10)$$

- The general expression of the Lie symmetry operator with $\varphi \equiv 1$:

$$U(x,y,t,u) = \frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u} \qquad (1.11)$$

- The symmetry invariance condition is given by the relation:

$$0 = U^{(2)}[u_t - A(x,y,t,u)u_{xy} - B(x,y,t,u)u_x u_y - C(x,y,t,u)u_{2x} - D(x,y,t,u)u_{2y} - $$
$$- E(x,y,t,u)u_y - F(x,y,t,u)u_x - G(x,y,t,u)]$$

- The previous relation has the equivalent expression:

$$0 = -A_t u_{xy} - B_t u_x u_y - C_t u_{2x} - D_t u_{2y} - E_t u_y - F_t u_x - G_t - A_x \xi u_{xy} - B_x \xi u_x u_y -$$
$$-C_x \xi u_{2x} - D_x \xi u_{2y} - E_x \xi u_y - F_x \xi u_x - G_x \xi - A_y \eta u_{xy} - B_y \eta u_x u_y - C_y \eta u_{2x} - D_y \eta u_{2y} -$$
$$-E_y \eta u_y - F_y \eta u_x - G_y \eta - A_u \phi u_{xy} - B_u \phi u_x u_y - C_u \phi u_{2x} - D_u \phi u_{2y} - E_u \phi u_y - F_u \phi u_x -$$
$$-G_u \phi + \phi^t - A\phi^{xy} - C\phi^{2x} - D\phi^{2y} - B\phi^x u_y - F\phi^x - B\phi^y u_x - E\phi^y$$

- Equating with zero the coefficients of various monomials in derivatives of u, we get 11 equations:

$$0 = \xi_u ;\ 0 = \eta_u ;\ 0 = B\eta_x - D\phi_{2u} ;\ 0 = B\xi_y - C\phi_{2u}$$

$$0 = A\eta_y - \eta A_y - A_u \phi + A\xi_x - \xi A_x + +2D\xi_y + 2C\eta_x - A_t$$
$$0 = A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u \phi - D_t$$
$$0 = -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x \xi - B_u \phi - B_y \eta$$

$$0 = -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x \xi - E_y \eta - E_u \phi + A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu}$$

$$0 = -\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x \xi - F_y \eta - F_u \phi$$
$$A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu}$$

$$0 = \phi_t + G\phi_u - F\phi_x - E\phi_y - G_t - G_x \xi - G_y \eta - G_u \phi$$
$$-A\phi_{xy} - C\phi_{2x} - D\phi_{2y}$$

APPLICATIONS

II.1. Diffusion of seismic waves.The 2D Ricci flow model

The *seismic waves* have been described as a combination between compression waves and shear waves (Adams–Williamson equation, 1923). In a more realistic picture, they can be seen as a nonlinear diffusion process. One can see the propagation of the seismic flow through the Earth's crust as generated by a gravitational interaction and describe it through *the Ricci flow equation* - a nonlinear parabolic equation obtained when the components of the metric tensor $g_{\alpha\beta}$ are deformed following the equation:

$$\frac{\partial}{\partial t} g_{\alpha\beta} = -R_{\alpha\beta}$$

$R_{\alpha\beta}$ is the Ricci tensor for the n -dimensional Riemann space. In the conformal gauge:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{2}\exp\{\Phi(X,Y,t)\}(dX^2 + dY^2)$$

The potential $\Phi(X,Y,t)$ satisfies the equation:

$$\frac{\partial}{\partial t} e^\Phi = \Delta\Phi$$

Introducing the field $u(x,y,t) = e^\Phi$ one obtains:

$$u_t = (\ln u)_{xy} = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \qquad (2.6)$$

Lie symmetries for 2D Ricci flow model:

- Lie symmetry operator:

$$X = \frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y} - u[\xi_x(x) + \eta_y(y)]\frac{\partial}{\partial u} \qquad (2.7)$$

- As $\{\xi, \eta\}$ = arbitrary functions, we deal with an infinite number of symmetry operators.

- The action of X can be split in various sectors, depending on the concrete form of $\{\xi, \eta\}$.

- On the *linear sector* with:

$$\varphi = 1,\ \xi = mx + c_1,\ \eta = vy + c_2,\ \phi = -(m+v)u \qquad (2.8)$$

$$X(x,y,t,u) = \frac{\partial}{\partial t} + (mx + c_1)\frac{\partial}{\partial x} + (vy + c_2)\frac{\partial}{\partial y} - (m+v)u\frac{\partial}{\partial u} \qquad (2.9)$$

- The Lie algebra is spanned by 4 independent symmetry operators:

$$V_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, V_2 = \frac{\partial}{\partial x}, V_3 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, V_4 = \frac{\partial}{\partial y} \qquad (2.10)$$

V_2, V_4 = space translations, V_1, V_3 = scaling transformations.

- The non-vanishing commutation relations: $[V_2, V_1] = V_2, [V_4, V_3] = V_4$

Invariant solutions for $2D$ Ricci flow

- **The operator** $V_2 + \alpha V_3$ has the characteristic equations: $\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{\alpha y} = \frac{du}{-\alpha u}$
- By integrating these equations we obtain 3 invariants: $I_1 = t,\ I_2 = ye^{-\alpha x},\ I_3 = yu$
- Similarity variables: $t = I_1, z = I_2 = ye^{-\alpha x}$ and $h(t,z) = I_3$
- Reduced equations:

$$u(t,x,y) = \frac{h(t,z)}{y};\ \ h_t h^2 - \alpha z^2 h h_{2z} - \alpha z^2 h_z^2 + \alpha z h h_z = 0$$

- The solution of the previous equation is:

$$h(t,z) = -\frac{1}{2}\left(r_3 t + \frac{r_2 r_3}{2r_1}\right)\left(-1 + \tanh^2\left(\frac{\sqrt{\alpha r_3}\,(r_4 - \ln z)}{2\alpha}\right)\right)$$

- The invariant solution corresponding to the operator $V_2 + \alpha V_3$ has the final form:

$$u(t,x,y) = -\frac{1}{2y}\left(r_3 t + \frac{r_2 r_3}{2r_1}\right)\left(-1 + \tanh^2\left(\frac{\sqrt{\alpha r_3}\,(r_4 - \alpha x + \ln y)}{2\alpha}\right)\right)$$

II.2 Vortices in self-organized systems. Chua circuit.

- The self-organized systems have the tendency to pass into highly structured and stable states. For example, the excitable media generate rotating spiral waves (vortex).
- Chua circuit allows to produce and to describe in real time the dynamics of such waves.

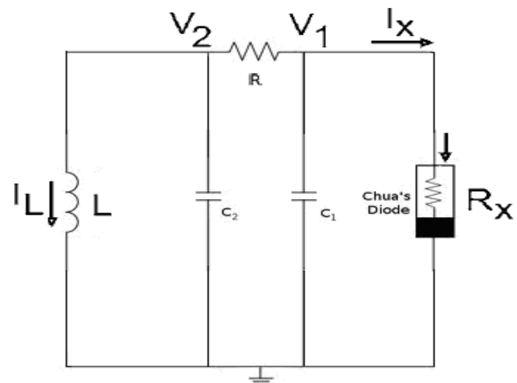


Fig. 1 Chua's electronic circuit

- Kirchoff ' s law for the circuit lead to a set of three differential equations of the form:

$$\begin{cases} \dot{x} = \alpha\left[(y-x)+f(x)\right] \\ \dot{y} = x - y - z \\ \dot{z} = -\beta y \end{cases}$$

- The function $f(x)$ describe the characteristic $I = f(V)$ of the nonlinear element know as Chua diode. Many types of nonlinearities were considered. In this paper we shall choose:

$$f(x) = \begin{cases} \sin x, & x \in \left[-\frac{\pi}{2k}, \frac{\pi}{2k}\right] \\ -1, & x < -\dfrac{\pi}{2k} \\ 1, & x > -\dfrac{\pi}{2k} \end{cases}$$

- This choice corresponds to a C^1 function on R whose graphic is presented in Fig.2.

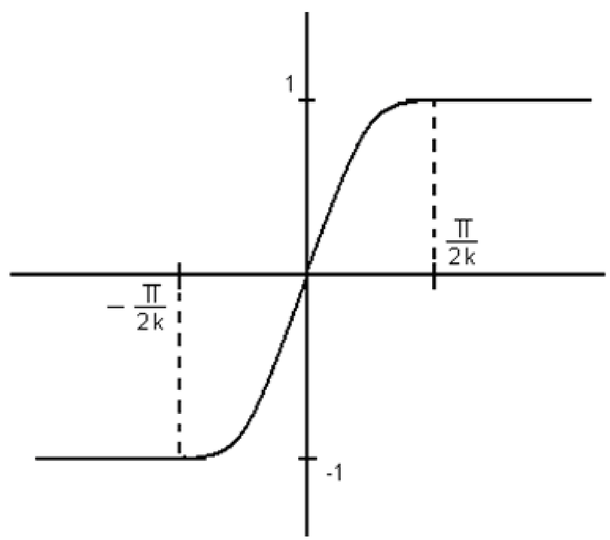


Fig. 2

We shall be interested in studying the chaotic behaviour and the regular regime corresponding to this circuit. In order to do this study, we shall compute the equilibrium points. They are given by:

$$\begin{cases} y - x + f(X) = 0 \\ x - y + z = 0 \\ -\beta y = 0 \end{cases} \Leftrightarrow \begin{matrix} x - f(x) = 0 \\ z = -x \\ y = 0 \end{matrix}$$

Three important cases can be identified:

Case 1 - A unique equilibrium point:

$$k \leq 1 \rightarrow P_1 = (0,0,0)$$

Case 2 - Three equilibrium points:

$$1 < k < \frac{\pi}{2} \rightarrow P_1 = (0,0,0); P_2 = (x_0, 0, -x_0); P_3 = (-x_0, 0, x_0)$$

Where x_0 denotes the positive solution for the following equation $x = \sin kx$

Case 3 – Three equilibrium points:

$$\frac{\pi}{2} < k \rightarrow P_1 = (0,0,0) \quad \overline{P}_2 = (1,0,-1); \overline{P}_3 = (-1,0,1)$$

Numerical simulations show that we do not have Hopf bifurcations, but regular and chaotic behavior can appear.

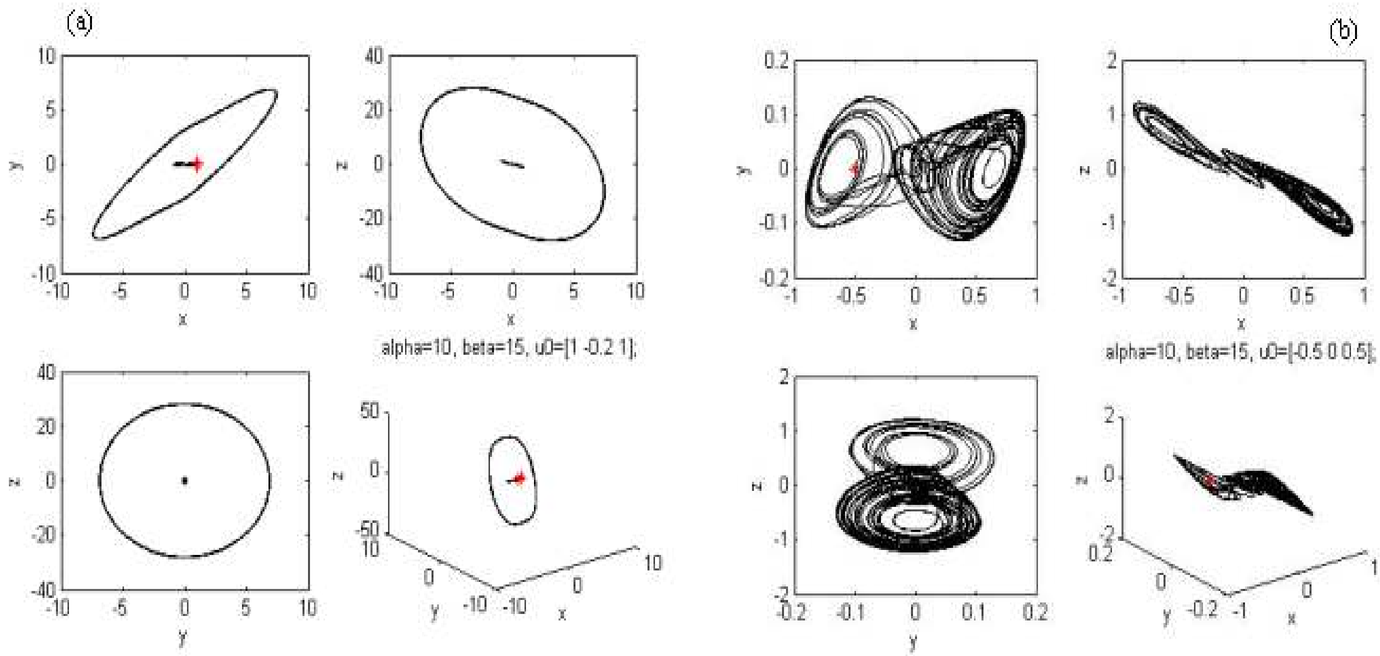
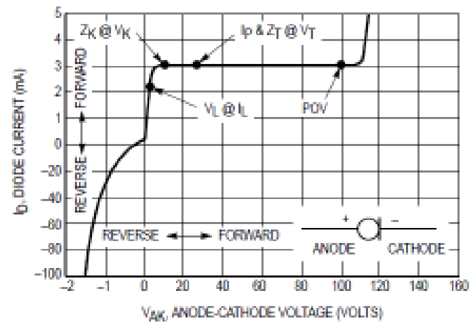


Fig. 3 (a) Limit cycle for Chua system in the case alpha = 10, beta =15, with initial conditions u_0 = [1, -0.2, 1] and (b) chaos for case alpha = 10, beta =15 and initial conditions u_0 = [-0.5, 0, 0.5].

Other chaotic circuit: *1N5314 diode*



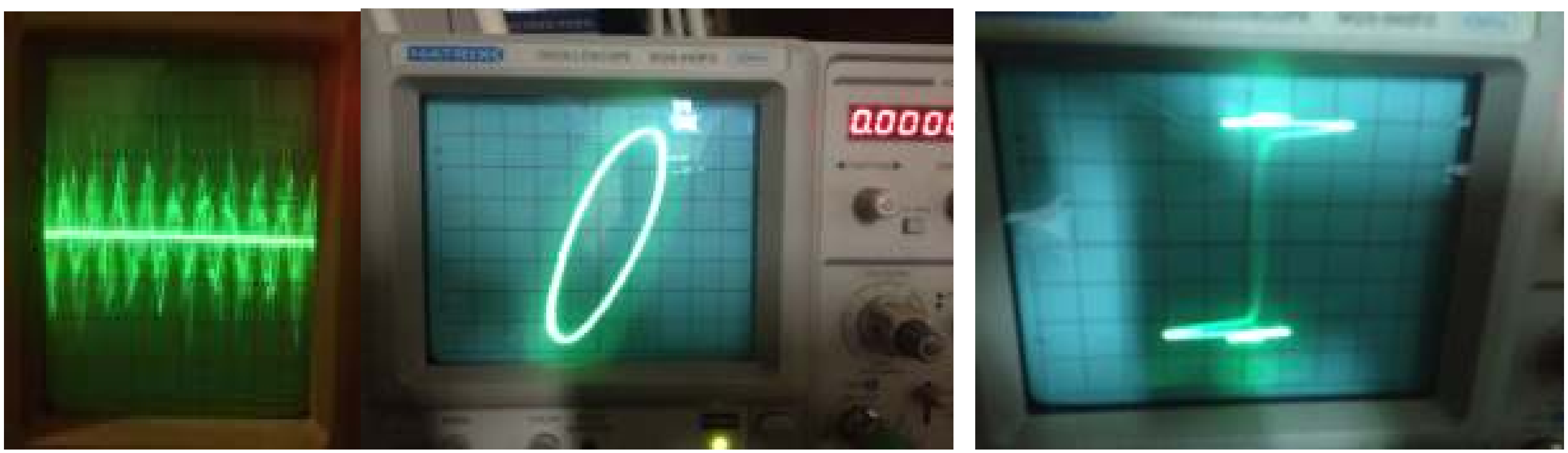


Fig. 4 (a) the dependence x=x(t); (b) limit cycle for 1N5314 diode for α = 10 β = 15 and u_0 = [-1, 0.2, 1]; (c) chaos double scroll attractor for α = 10 β = 15 and u_0 = [-0.5, 0.2, 0.5];

CONCLUSIONS

- **Nonlinear dynamics deals with evolutionary systems described through nonlinear differential equations (in the continuous case).**
- **Description of the evolution of the system supposes to find solutions of the equations.**
- **For nonlinear equations there are not clear integration rules. Sometimes it is enough to decide if the system is integrable or not.**
- **The integrable systems could have various classes of solutions: stable/unstable, regular (periodic)/chaotic.**
- **Chaos is deterministic.**
- **The nonlinearities of the dynamical systems are expressed through continuous functions. The natural hazards involve discontinuities/ critically changes of the state.**

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Thank you for your attention!